Proof Nets for Classical Logic

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Abstract. This paper introduces a notion of proof net for classical logic, provides a static correctness condition for these nets, and analyses the connection between nets and conventional sequent calculus. The main surprise of the paper is that there are no surprises at the static level. Subsequent work reveals that there are few at the dynamic either.

Introduction

This paper is a first step towards adapting the notion of proof nets, first introduced by Girard for Linear Logic in his [4], to standard Classical Logic. This paper deals mainly with the static aspects of the extension. The dynamical aspects await a companion paper. The surprise is how much of the conventional treatment extends without significant difficulty.

The structural proof theory of Classical Logic is problematic, particularly in comparison with that for Intuitionistic Logic. For intuitionistic logic there are two standard forms of structural proof system: natural deduction and the sequent calculus. Both these forms have a good correspondence with functional type theories, natural deduction with a conventional form, and sequent calculus with one with explicit substitution [7, 10, 11, 15]. The systems relate well to one another. There is a structural translation of sequent proofs to natural deduction ones ([3, 16, 9]) that by and large behaves well with respect to the different notions of reduction (there is one small glitch, with a standard cut reduction for disjunction corresponding to a non-standard form of distribution in natural deduction [14]) formal reduction properties of the systems relate well to one another. The formal properties of the reduction systems are good, with that for natural deduction being confluent and terminating. This allows the semantic identification of a proof with its normal form. Finally, the logical rules of natural deduction are conveniently divided into introduction and elimination rules, with the elimination rules expressing the completeness of the introductions ([8, 13]).

By contrast, when we look at Classical Logic, the first question is what we mean by "Classical Natural Deduction". The standard answer (dating back to Gentzen [3] is to extend Intuitionistic Natural Deduction with the rule of double negation elimination:

$$\frac{\neg\neg A}{A}\ \neg\neg E$$

But this is neither an introduction nor an elimination rule for the actual negation connective, whose properties are therefore no longer completely described by its introduction rules. In fact, no system in introduction/elimination format is known for Classical Logic. This has the effect of somewhat complicating the translation from sequent calculus to natural deduction.

Classical Sequent Calculus is, however, quite definitive, and has remained remarkably stable since Gentzen. The main developments have been the investigation of tweaks to do with the placing of structural rules, and an understanding, inspired by Girard, of the different implications of choosing additive or multiplicative formulations of the rules. However, reduction for classical sequent calculus is non-deterministic and non-terminating. We expect the former at least to feed through to classical natural deduction. This means that we can no longer expect simply to equate a classical proof with its normal form. Furthermore, there may be points at which we have to take evaluation order seriously. Current treatments of Classical Natural Deduction relate it to languages with control operators like `call/cc`. We can give perfectly sensible semantic treatments of non-deterministic versions of programming languages with these features, so there must be some hope for Classical Logic too. From this respect the well-known problems with semantics for Classical Logic are not relevant: they are based on the assumption that proof reduction preserves equality.

This then raises the question of what definition of equality might be sensible for classical proof. We cannot hope for a definitive answer at this stage, but we can begin to map out the territory. The standard approach for programming languages is to use a contextual equivalence: things can only be the same if they appear the same in any context. It makes sense at least initially to say that distinct normal forms are different. Putting these together we get: two proofs are different if, when we plug them into some context, one of the complete proofs produces a normal form that cannot be reached from the other.

Christian Urban, in joint work with Martin Hyland and myself, has investigated some of the consequences of this definition when the proof system involved is a form of classical sequent calculus under cut reduction. He has produced examples to show that the proofs have a worse behaviour than we would like. For example, in the intuitionistic case, we can define a category in which objects are propositions and morphisms observational equivalence classes of proofs (composition is given by cut and identities by axioms). In this category conjunction is cartesian product. However, when we construct the analogue for classical logic, conjunction is no longer even functorial. Although we can construct the "conjunction" of two proofs, the conjunction of two axioms does not behave as an identity: when we cut against it we can reach additional normal forms. This says that conjunction does not preserve identities. All is not completely lost. It appears that we may be able to recover some kind of structure by focusing on a subset of good "linear" proofs, and characterising good behaviour as relative to those. Nevertheless the situation is hardly pleasant. In trying to explain the behaviour of these counterexamples we were led to look at more diagrammatic representations of the proofs. These became the proof nets of this paper. They are not however an original discovery. They are simply an adaptation of the proof nets of Girard. However, our concerns are different from Girard's. We are seeking a system which has a tight relationship with the existing system of classical sequent calculus, not a system that has good behaviour in Girard's sense!

Proof nets were introduced for Classical Linear Logic by Jean-Yves Girard in his original paper [4]. At one point Girard describes them as "a classical natural deduction". He does not intend by this that they are related to a form of classical natural deduction proof system. They are not. They arise out of a sequent system in a way that we shall make clear later. Rather he intends that they can be used as a kind of term calculus for computation, in the same way that natural deduction proofs give lambda terms that can be computed with.

Existing work on proof-nets is mainly for flavours of Linear Logic. One of the lessons there is that the theory is very smooth for the multiplicative connectives, but more problematic for additives, which require "boxes" to indicate subproofs [6]. We therefore adopt a multiplicative presentation of classical logic. The remaining problems are therefore the structural rules of contraction and weakening (and contraction is known not to be a problem). Moreover the existing work is adapted to a one-sided presentation of sequent calculus. This has the consequence that negation and implication cannot be treated as primitive connectives (the rules involve both left and right-sided information). Instead they have to be treated as inductively defined operations on propositions. This is a perfectly reasonable thing to do, and it leads to some technical simplifications, but it requires justification, and that justification is best given either in terms of a relationship between the dynamics of two-sided and single-sided proof systems, or between two-sided and single-sided proof-nets. In this paper we present one form of two-sided net.

In his seminal paper on classical logic [5], Jean-Yves Girard discusses the possibility of adapting the notion of proof-net so as to get a notion of classical proof net. He indicates very briefly that it suffices to introduce n-ary contraction links, which are to be treated as an n-ary "par". He then goes on to say that "so far as we exclude 0-ary contraction links (i.e. weakening links) then there is a straightforward correctness condition... If 0-ary links are considered no non-trivial condition is known, but the notion still makes sense." From the narrowest technical perspective, the purpose of this paper is to provide such a condition. But there is a bit more than this. The Danos-Regnier switching acyclicity technique for establishing correctness depends only on the format of the rules involved. It extends trivially to contraction because, from the point of view of the resulting graphs, these rules have the same format as rules already in the calculus. Weakening is different, and we shall extend the class of proof rules for which the Danos-Regnier technique can be used.

Of course, the dynamics of the nets is equally important. It is not difficult to define a dynamics for the nets we give here, and to prove results relating that dynamics to classical cut reduction. We defer the details to [12].

In [5], Girard also gives arguments to show that in the presence of contraction or weakening it is impossible to have a symmetric form of cut elimination.

He argues that for a particular proof involving propositions over atomic a and b, the group $\mathsf{S} = \mathsf{C}_2 \times \mathsf{C}_2$ generated by "interchange a and $\neg a$" and "interchange b and $\neg b$" acts on the proof net, and so should act on its normal form. However, Girard seeks a deterministic elimination in which the normal forms correspond to standard proofs in normal form. Girard indeed shows that this is not possible. But in the standard non-deterministic reduction, the group also acts on the set of normal forms. In Girard's example this essentially amounts to a presentation of the group as a semi-direct product (of the group of permutations of the normal forms by the automorphisms of a single normal form). This argument will therefore not present a problem to us.

The paper begins with a discussion of classical sequent calculus, and then moves on to the definition of proof net as a graphical structure that captures the local information in the sequent proof. We then prove that any proof arises from a sequent proof, and characterise the proofs that give rise to a given net in terms of permutations of rules. We conclude with a brief discussion of reduction, leaving the proofs of properties to a subsequent paper.

I would like to thank Gian-Luigi Bellin, Martin Hyland, and Christian Urban for helpful discussions, and David Pym for his encouragement and patience during the writing of this paper.

1 Graphical representation of sequent calculus

The sequent calculus only allows us to apply one rule to a sequent at a time. It therefore forces us to order independent applications of proof rules. To take a trivial example, in the proof fragment

$$\dfrac{\dfrac{\dfrac{\cdots}{A, B, C, D \vdash (A \wedge B) \wedge (C \wedge D)}}{A, B, C \wedge D \vdash (A \wedge B) \wedge (C \wedge D)}\,\wedge\mathrm{L}}{A \wedge B, C \wedge D \vdash (A \wedge B) \wedge (C \wedge D)}\,\wedge\mathrm{L}$$

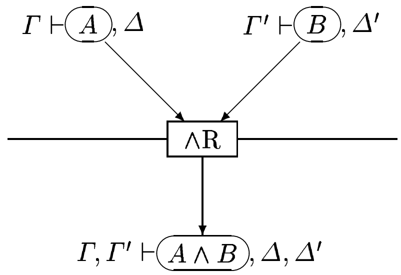
the two applications of the $\wedge$L rule are intuitively completely independent, and there seems no sensible reason why we should distinguish this proof from

$$\dfrac{\dfrac{\dfrac{\cdots}{A, B, C, D \vdash (A \wedge B) \wedge (C \wedge D)}}{A \wedge B, C, D \vdash (A \wedge B) \wedge (C \wedge D)}\,\wedge\mathrm{L}}{A \wedge B, C \wedge D \vdash (A \wedge B) \wedge (C \wedge D)}\,\wedge\mathrm{L}$$

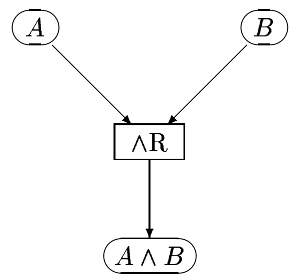
The same phenomenon can arise with formulae on different sides of the turnstile, as in

$$\dfrac{\dfrac{\dfrac{\cdots}{A, B \vdash C, D}}{A, B \vdash C \vee D}\,\vee\mathrm{R}}{A \wedge B \vdash C \vee D}\,\wedge\mathrm{L}$$

This naturally leads us to try to produce a more graphical notation which mirrors the sequent calculus, but in which the sequent as a whole disappears and is replaced by links only between the formulae actually involved. Take the $\wedge$R rule as an example.



Removing the rule and the non-participatory formulae we get a figure



in which we have put (occurrences of) formulae in circles, and (the occurrence of) the rule in a box. We have also put directions on the links, with arrows leading from hypotheses to the rule, and from the rule to its conclusions. We get one such figure for each rule of the calculus (except for exchange, where we get none). We must of course distinguish between left and right rules for the connectives. Less obviously we must also distinguish somehow between occurrences of formulae to the left and right of the turnstile. Formulae to the left of the turnstile should not suddenly turn into formulae to the right! The figures for the different proof rules are given figs. 1 and 2. They are written to follow the pattern of the rules, with hypotheses at the top and conclusions at the bottom. Directions on the arcs follow the construction of the proof, so that arcs are directed from premisses to the rule, and from the rule to conclusions. An alternative would be to direct arcs using a notion of information flow.[1] However, the results we give are

[1] If a left formula occurs in a premiss to the rule, then the arc is directed from the rule to the formula, and if it occurs in the conclusion, then the arc is directed from the formula to the rule. Dually, if a right formula occurs in the premiss, then the arc goes from the formula to the rule, and if it occurs in the conclusion, then the arc goes from the rule to the formula.

about construction of proofs, not flow of information, and this latter representation, although intuitively attractive, unfortunately complicates the presentation a little, notably by demanding a splitting of the figures for weakening.

Proof-nets are constructed by pasting these figures together at the formulae. There obviously has to be agreement on the formula, and whether it is to the left or right of the turnstile. Moreover, each formula (occurrence) can have at most one arc leading into it, and at most one leading out. Thus we can in effect regard the formula either as a label on a wire leading between two rules, or as a label on a (directional) socket. In a complete structure, the formulae are produced from axioms by the application of rules, and so every formula must have a (unique) arc leading to it. Had we followed the information-theoretic direction of arcs there would be a similar but slightly more complex condition. Left formulae are regarded as inputs, and right as outputs, thus if a formula has a single arc joined to it, then that must be outgoing if it is a left formula, and incoming if a right.

We can now give a more formal definition:

Definition 1. *A* proof structure *is a finite bipartite directional graph whose two families of nodes are labelled as follows:*

[Family 1] *labelled by one of the sequent proof rules*
[Family 2] *labelled by a formula, together with the information Left or Right.*

The graph is subject to the following additional constraints:

1. *the graph surrounding each rule node is given uniquely as an instance of the corresponding figure*
2. *each propositional node has a unique incoming and at most one outgoing arc*

There is some ambiguity in the phrase "the graph surrounding each rule node is given uniquely as an instance of the corresponding figure". We intend that this mapping is given as part of the structure of the graph. In most instances only one such mapping will be possible, but we will wish to distinguish the two inputs to, say, an ($\wedge$R) even when they are instances of the same proposition. When we come to treat equality of nets, however, contraction will be treated as symmetric.

Definition 2. *A* substructure *of a proof structure is a subgraph which is also a proof structure.*

Suppose we have a subgraph of a proof structure. Constraint 1 says that if it contains a rule node, then it must also contain the surrounding propositional nodes. Constraint 2, on the other hand, does some real work. It says that if the proof contains a propositional node, then it must also contain the rule generating it, the hypotheses for that node, and so on up. There is therefore an inductive closure process. It is easy to see by inspecting the figures that this process can only terminate in axioms (it may of course in principle also generate cycles).

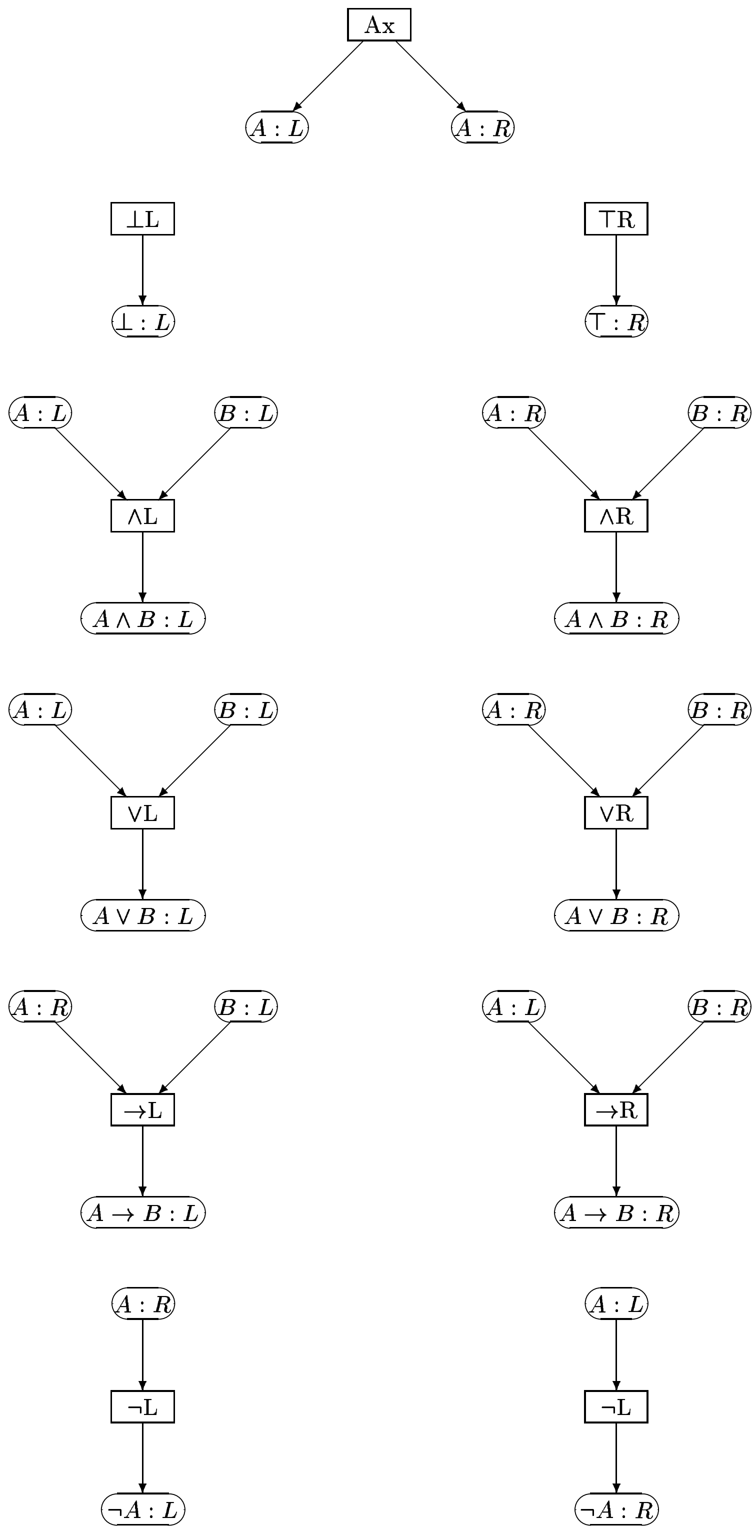


Fig. 1. Figures for sequent calculus

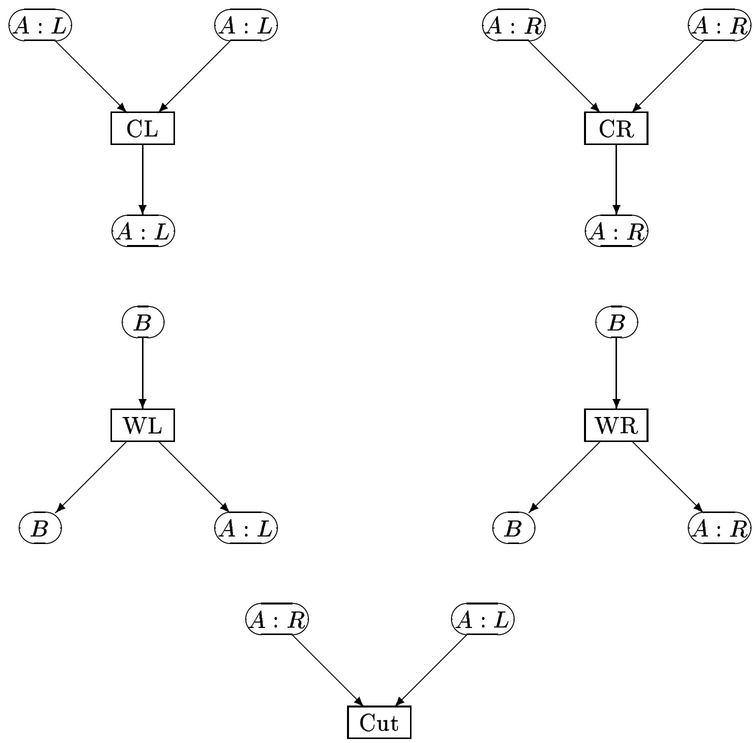


Fig. 2. Figures for sequent calculus, cont'd.

$$\frac{}{A \vdash A}\ \text{Ax}$$

$$\frac{}{\vdash \top}\ \top\text{R}$$

$$\frac{}{\bot \vdash}\ \bot\text{L}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge\text{L} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\ \wedge\text{R}$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}\ \vee\text{L} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee\text{R}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \rightarrow B \vdash \Delta, \Delta'}\ \rightarrow\text{L} \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\ \rightarrow\text{R}$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg\text{L} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg\text{R}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{CL} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \text{CR}$$

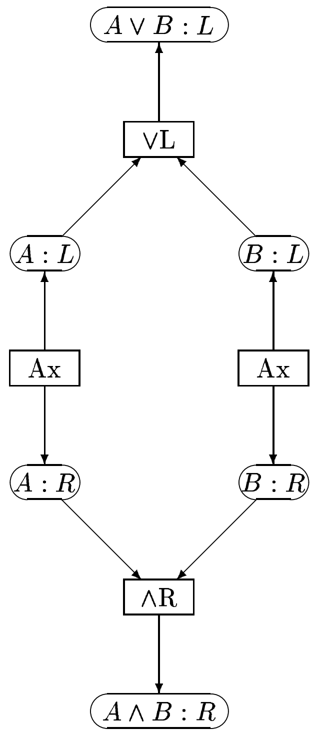
$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{WL} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \text{WR}$$

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\ \text{EL} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\ \text{ER}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Cut}$$

Fig. 3. Proof rules for classical sequent calculus (multiplicative form)

However, we still have structures which do not represent valid proofs, for example



which purports to be a proof of the invalid $A \vee B \vdash A \wedge B$.

These structures are eliminated by using a technique due to Danos and Regnier [2].

Definition 3. *A* (Danos-Regnier) switching σ *is the choice of one of the hypotheses for each node of the following forms: [$\wedge$L], [$\vee$R], [$\rightarrow$R], [CL], [CR]. We shall say that the remaining nodes are* unswitched.

The purpose of a switching is to generate a graph.

Definition 4. *Let S be a proof structure and σ a switching on it. Then the* (Danos-Regnier) *graph of σ, $DR(\sigma, S)$, is the following undirected graph:*

- *its vertices are the propositional vertices of S,*
- *its edges join conclusions of rule nodes to hypotheses as follows. If the rule node is unswitched, then each conclusion is joined to each hypothesis. If the rule node is switched then the conclusion is joined only to the hypothesis chosen by σ. The exceptions are axioms and cut, where each of the two formulae is joined to the other.*

Definition 5. *A proof structure S is a* proof net *if for each switching σ of S the Danos-Regnier graph of σ, $DR(\sigma, S)$, is connected and acyclic (as an undirected graph). A* subnet *of a proof net is a substructure which is also a proof net.*

The first significant result of the paper will be that proof nets represent proofs. The proof follows the traditional proof for MLL as in, say, [1]. The crucial fact to notice is that the Danos-Regnier technique is independent of the particular logic involved, and depends only on the format of the rules involved. The more interesting of these are the rules for binary connectives, which fall into two categories: ones in which the formulae linked by the connective come from a single subproof, and ones in which they come from two different ones, as in

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \to \psi, \Delta} \to\mathrm{R} \qquad \frac{\Gamma \vdash \phi, \Delta \quad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', \phi \to \psi \vdash \Delta, \Delta'} \to\mathrm{L}$$

In the Danos-Regnier technique we build an undirected graph on the propositional nodes of a proof-net. The soundness of the technique depends on the fact that if the proof-net comes from a proof then this graph is simply-connected. This requirement forces the switching condition. In the ($\to$L) rule, ϕ and ψ come from separate subproofs, so are not connected, and so $\phi \to \psi$ must be joined to both, or the graph will be disconnected. In other words, the rule must be unswitched. On the other hand, in the ($\to$R) rule, ϕ and ψ come from the same subproof, so are already connected. We must join $\phi \to \psi$ to only one, or the resulting graph will contain a cycle, and the rule must be switched. From this perspective, the contraction rules have the same form as ($\to$R). In

$$\frac{\Gamma, \phi, \phi \vdash \Delta}{\Gamma, \phi \vdash \Delta} \mathrm{CL} \qquad \frac{\Gamma \vdash \phi, \phi, \Delta}{\Gamma \vdash \phi, \Delta} \mathrm{CR}$$

both occurrences of ϕ come from the same subproof, so the rules must be switched.

But the weakening rules have a new form: they have apparently no hypothesis and a single conclusion. Nevertheless, a similar analysis tell us what to do with them. If we are to use the existing technology, which characterises proof nets in terms of simply connected Danos-Regnier graphs, then the formula introduced by a weakening rule must be linked in to the pre-existing proof. There *is* a pre-existing proof since the empty sequent " $\vdash$ " is not provable. In principle we could link the new formula in anywhere, but we may as well use one of the last formulae remaining in the sequent to be introduced.

Similarly, when we have an axiom, the two consequences of the axiom must be linked, and when we have a cut, the two cut formulae must be (the link, as it were, passes through the rule node).

The structures we are using are to some extent adapted to proof search. There is a validity condition for nets that represent an incomplete sequent proof constructed by backwards chaining from the conclusion, and so with unproved sequents at the top as assumptions. When we construct the Danos-Regnier graph, we link all the formulae in any unproved sequent to an artificial top node representing the hypothesis for that sequent.

2 Translating sequent proofs to nets

The translation of a sequent proof into a proof net can be formally defined by induction on the structure of the proof. The sequent being proved can be identified from the net via the set of propositional nodes at the edge of the proof in the sense of being connected by only one arc. By definition 1(2) this arc must be an incoming one. These terminal nodes are called *doors* (see definition 6). The doors of the structure give a multiset of propositions labelled left and right, and hence a sequent. When we translate a proof into a net, we get a correspondence between the propositions in the sequent being proved and the doors of the net.

In order to translate a sequent proof into a proof net we first translate the immediate subproofs. We then attach a figure corresponding to the final rule application at the appropriate points. For all the rules except weakening these points are uniquely determined, at least up to ambiguities in the sequent calculus itself about which occurrence of a formula in the hypotheses corresponds to which in the conclusion. For weakening we simply attach the figure at one of the doors. Any will do.

The derivation of a proof net from a sequent proof is thus non-deterministic, but only up to movements of weakening links.

3 Different notions of proof net

The structures we have defined are a little cluttered. It is possible to achieve the same result with less structure. For example if we take the picture given above seriously, then we only really need a graph based on one of the two families of nodes. For someone brought up in the tradition of Curry-Howard and syntactic encodings of natural deduction proofs it is natural to take the rule nodes as basic and treat the propositions as mere labels on arcs between rule nodes. On the other hand it is also possible to take the propositional nodes as basic, and treat the rule nodes as multilinks joining the propositional nodes. This is the approach adopted by Girard in his proof nets for linear logic. In terms of informational content these approaches are all equivalent, and one representation can be translated without difficulty into another.

However, there is one further difference in Girard's approach. Girard formalises a one-sided calculus. A consequence is that negation becomes a defined operation, rather than a connective, and, at least on the surface, implication disappears.

4 Some basic properties

The notion of switching is fundamental to the global correctness criterion. It is tempting to view it as giving an account of some form of dataflow (as perhaps in Girard's trip criterion), but it seems that it is clearer when viewed as a purely formal consequence of the structure of the proof rules. This structure determines whether a rule is switched or not. In general, for a rule of the form

$$\frac{\Gamma_0 \vdash \Delta_0 \quad \ldots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

a switching consists of the choice of one of the active formulae of each of the $\Gamma_i \vdash \Delta_i$. The Danos-Regnier graph is constructed by joining together the chosen formulae and all of the active formulae in $\Gamma \vdash \Delta$ using a connected acyclic piece of graph (adding an extra vertex if necessary).

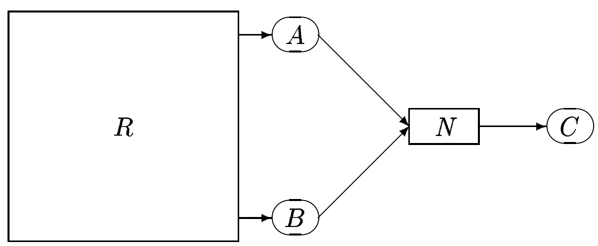
Although the techniques work for any system that has rules of this form, we shall concentrate on the concrete case of classical logic. We begin with some simple lemmata.

Lemma 1. *Let R_1 and R_2 be subnets of the proof net R, then*

1. *$R_1 \cup R_2$ is a subnet iff $R_1 \cap R_2$ is non-empty*
2. *if $R_1 \cap R_2$ is non-empty, then it is a subnet.*

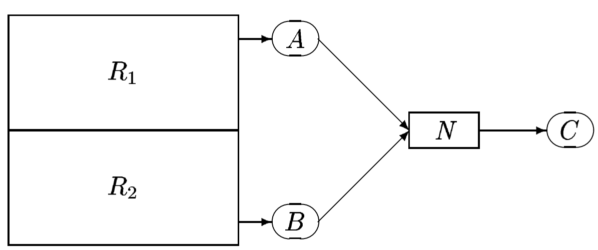
Proof. Since R is a net, then its Danos-Regnier graph contains no cycles, and hence nor does that of any substructure. So a substructure is a subnet iff its Danos-Regnier graph is non-empty and connected. This will be the case for a union iff the intersection is non-empty. For an intersection, we must first rule out the case of an empty proof. But if the intersection is non-empty, take any switching and extend it to a switching of R. Take any two nodes of $R_1 \cap R_2$, then they are connected by a path in the Danos-Regnier graph of R. Since R_1 and R_2 are both nets, this path lives in their Danos-Regnier graphs, and hence in the intersection.

Proposition 1. *1. Let*



be a proof structure with the switched node N at the edge. Then this is a proof net if and only if R is a proof net.

2. *Similarly for proof structures with weakening and negation at the edge.*
3. *Let*



be a substructure of the proof structure R, where the node N is unswitched. Suppose R_1 and R_2 are themselves proof nets. Then the graph given is a proof net if and only if $R_1 \cap R_2 = \emptyset$.

4. *Similarly for [Ax] and [Cut].*

Theorem 1. *The structure derived from a sequent proof is a proof net.*

Definition 6. *Let S be a proof structure. Then a* door *of S is a propositional node which is at the edge of S in the sense of being connected by only one arc, necessarily incoming.*

Definition 7. *Let R be a proof net.*

1. *The* territory *of a set Σ of propositional nodes, $t(\Sigma)$ is the smallest subnet of R containing Σ (not necessarily as doors).*
2. *If A is a propositional node, its* kingdom *$k(A)$ (resp.* empire *$e(A)$) is the smallest (resp. largest) subnet of R with A as a door.*
3. *If A and B are propositional nodes, then $A \ll B$ iff $A \in k(B)$.*

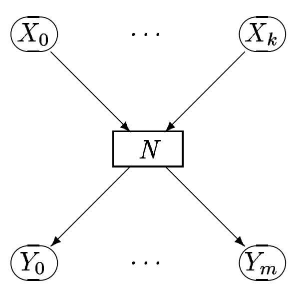
By lemma 1, $\mathrm{t}(\Sigma)$ always exists, and $\mathrm{k}(A)$ and $\mathrm{e}(A)$ exist if there is any subnet which has A as a door. We shall show that $\mathrm{e}(A)$ exists, and give a construction of it. Note that if σ is a switching then the restriction of $\mathrm{DR}(\sigma, S)$ to $\mathrm{e}(A)$ is connected and, since A is a door, does not contain links with A as a hypothesis. It follows that $\mathrm{e}(A)$ is a subset of any $\sigma(S, A)$ defined below.

Definition 8. *Let A be a propositional node in a proof structure S, and σ a switching for S. Let $\sigma(S, A)$ be the subgraph of $DR(\sigma, S)$ defined as follows. If A is the premise of a rule node, then remove all links in $DR(\sigma, S)$ which pass from A to a corresponding conclusion (if A is the premise of a cut, remove the link to the other cut formula), and take the component of A in the result. If A is not the premise of a rule node, then take $DR(\sigma, S)$.*

We will show that $\mathrm{e}(A)$ is the intersection of all these sets. We will also show that it can be defined inductively.

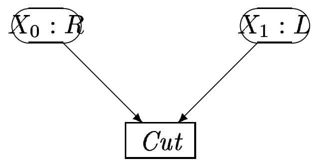
Definition 9. *Given a propositional node A in the proof structure S, then we say that a set E of propositional nodes is* $\mathrm{e}(A)$-closed *iff*

1. *A is in E*
2. *E is closed under hypotheses (i.e. upwards-closed in S)*
3. *if X_0 and X_1 are the conclusions of an axiom rule, and X_0 is in E, then so is X_1, and vice versa*
4. *if*



is an unswitched node, and for any i, $X_i \neq A$ *and* $X_i \in E$, *then for each* j, $Y_j \in E$.

5. *if, in the above,* N *is a switched node, and for all* i, $X_i \neq A$ *and* $X_i \in E$, *then for each* j, $Y_j \in E$.
6. *if*



is a cut node and for some i, $X_i \neq A$ *and* $X_i \in E$, *then* $X_{1-i} \in E$.

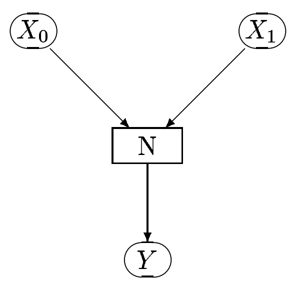
Proposition 2. *Let* R *be a proof net containing propositional node* A. *Then the empire of* A *exists and is given by the two following constructions:*

(a). $E_a = \bigcap_\sigma \sigma(R, A)$, *taken over all switchings* σ *of* R.
(b). E_b *is the smallest* e(A)*-closed subset of* R.

Proof. ($E_b \subseteq E_a$): We show that E_a is e(A)-closed.

In fact, if σ is a switching, then $\sigma(R, A)$ satisfies conditions 1 and 3-6 of the definition of e(A)-closed set. Moreover, it satisfies condition 2 for unswitched nodes. Sets which satisfy these conditions are stable under intersection, and hence E_a also satisfies them.

It remains to show that E_a is closed under hypotheses of switched nodes. Suppose

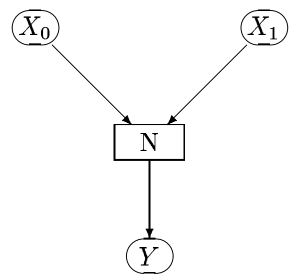


is a switched node, and that $Y \in E_a$, but that X_1, say, is not.

Note that X_0 cannot be A. If it were, consider any switching σ linking Y to $X_0 = A$. Then the unique path from A to Y in DR(σ, R) consists of the single link that is eliminated in the construction of $\sigma(R, A)$. Hence Y can not be in $\sigma(R, A)$.

Now pick a switching σ such that X_1 is not in $\sigma(R, A)$. Since $Y \in \sigma(R, A)$, σ must link Y to X_0. Moreover, the path from A to X_1 must contain, and hence start with, a link eliminated in $\sigma(R, A)$ from A to a conclusion B. The path from B to X_1 cannot pass through Y, since Y and X_1 are in different components of $\sigma(R, A)$, and hence cannot use the link from Y to X_0 chosen by

the switching. This therefore remains a valid path in $\mathrm{DR}(\sigma', R)$, where σ' is the switching identical to σ except that Y is now linked to X_1. Now in $\mathrm{DR}(\sigma', R)$ we have a path from A to B to X_1 to Y staying outside $\sigma'(R, A)$, and a path A to Y in $\sigma'(R, A)$. We therefore have a cycle in $\mathrm{DR}(\sigma', R)$, contradicting the proof-net condition for R.
($E_a \subseteq E_b$): Let σ be a *principal switching for* A, i.e. a switching such that if



is a switched node, and X_i is in E_b but Y is not, then σ chooses X_{i-1} (with the exception that if one of the hypotheses is A, then σ should choose it). We can think of a principal switching as attempting to make $\sigma(R, A)$ as small as possible at the possible boundary nodes. We show that $\sigma(R, A) \subseteq E_b$.

Suppose a propositional node W is in $\sigma(R, A)$ but not E_b. Consider the path from A to W in $\sigma(R, A)$. This must exit E_b at some point. That cannot be at A since at A we must move to a hypothesis for A, and E_b is closed under hypotheses. Similarly it cannot be at any other point where we are moving from a formula to a hypothesis. Thus it must be at some point where we are moving from a formula to a conclusion, or across a cut. The closure conditions on E_b imply it cannot be at a cut, therefore it must be at some move from premise to conclusion. By condition 4 this cannot be at an unswitched node. But it cannot be at a switched node either, since the principal switching sets the switch in the wrong direction.

We have now shown that the two definitions, E_a and E_b are equivalent.
(A is a door of E_a): Suppose A is a premise of some rule with conclusion B. Let σ be any switching that links B to A. Then B is not in $\sigma(R, A)$ and hence not in E_a. Similarly, if A is the premise of a cut, then the other premise cannot be in $\sigma(R, A)$ for any switching σ.
($E_a = E_b$ is a subnet): Let σ be a switching of $E_a = E_b$. Then, since the criterion for a principal switching refers only to nodes not in E_b we can always extend σ to a principal switching on R. $\mathrm{DR}(\sigma, E_a)$ is then $\sigma(R, A)$ which is necessarily connected and acyclic.
($E_a = E_b$ is the largest subnet with A as a door): Any such subnet must be contained in each $\sigma(R, A)$.

Corollary 1. *If R is a proof net, and A any propositional node in it, then $e(A)$ and $k(A)$ both exist.*

The following technical lemma gives a nesting property for subnets that is crucial in the proof that nets represent proofs.

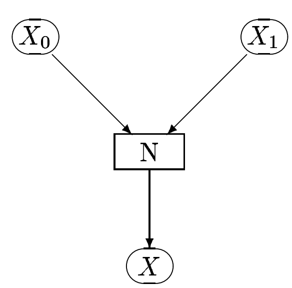
Lemma 2. *Suppose R is a proof-net in which N_0 is a rule node which has the propositional node A as one of its hypotheses and C as one of its conclusions, and N_1 is a rule node which has B as one of its hypotheses and D as one of its conclusions. Suppose moreover that $A \neq B$ and that $B \in e(A)$, then $D \in e(A)$ iff $C \notin k(D)$.*

Proof. Suppose $D \in \mathrm{e}(A)$, then $\mathrm{e}(A)$ is a subnet containing D. Hence $\mathrm{k}(D) \subseteq \mathrm{e}(A)$. But $C \notin \mathrm{e}(A)$, since A is a door of $\mathrm{e}(A)$. Therefore $C \notin \mathrm{k}(D)$. Conversely, $\mathrm{k}(D)$ and $\mathrm{e}(A)$ are both subnets containing B. It follows that their union is also a subnet (since they intersect). Now, if $C \notin \mathrm{k}(D)$, then $\mathrm{k}(D)$ does not contain the rule node that links A and C, and hence A is still a door of the union. It follows that $\mathrm{k}(D) \cup \mathrm{e}(A) = \mathrm{e}(A)$, and hence $\mathrm{k}(D) \subseteq \mathrm{e}(A)$. But $D \in \mathrm{k}(D)$, and hence $D \in \mathrm{e}(A)$.

Lemma 3. *Let R be a proof-net and X and Y propositional nodes in it. Then $k(X) = k(Y)$ iff X and Y are conclusions of the same rule node.*

Proof. If X and Y are conclusions of the same rule (either an axiom or a weakening if $X \neq Y$), then any substructure which contains X also contains Y, and conversely. Hence the kingdoms are equal.

Conversely, suppose X and Y are not conclusions of the same rule, but that $Y \in \mathrm{k}(X)$. Consider the rule which has X as conclusion. If this is switched, or has a single hypothesis, then it can be removed from $\mathrm{k}(X)$ leaving a subnet which still contains Y. It follows that $X \notin \mathrm{k}(Y)$. If the rule which has X as conclusion has multiple premises and is unswitched:



then $\mathrm{k}(X) = \{X\} \cup \mathrm{k}(X_0) \cup \mathrm{k}(X_1)$ and Y must be in either $\mathrm{k}(X_0)$ or $\mathrm{k}(X_1)$. But in neither case is X then in $\mathrm{k}(Y)$.

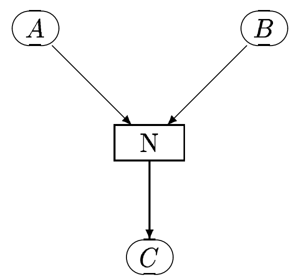
Lemma 4. *The relation $X \ll Y$ iff $X \in k(Y)$ is a partial ordering of propositional nodes which are conclusions of single-conclusion rules.*

Proof. Immediate from the preceding lemma.

This ordering gives us enough information to show how to present a proof-net as the image of a sequent proof. We extend the ordering relation to non-cut rule nodes by $M \ll N$ iff $X \ll Y$ where X and Y are conclusions of M and N respectively, and to cut nodes by treating them as unswitched nodes which generate a special proposition "cut" as conclusion. Notice that this is a genuine partial order on rule nodes.

Theorem 2. *If R is a proof-net, then R can be generated as the image of a sequent proof (sequentialised).*

Proof. The proof is by induction on the number of rule nodes in R. If R contains only a single node, then it must be an axiom. If any final node is switched or has only a single premise, then it can be deleted to leave a smaller net R', which can be sequentialised. It is then trivial to use the sequentialisation of R' to construct a sequentialisation of R. Suppose, then that all the final nodes are multiple premise unswitched nodes (or cut nodes). Choose one which is maximal with respect to $\ll$, say



Since N is unswitched and $\mathrm{e}(A)$ and $\mathrm{e}(B)$ are subnets, they must be disjoint. However $R = \{C\} \cup \mathrm{e}(A) \cup \mathrm{e}(B)$. Suppose not, then there must be some link with premise D and conclusion E such that D is in $\mathrm{e}(A)$, say, but E is not. It follows from lemma 2 that $C \in \mathrm{k}(E)$. Now let M be any final rule node below E (i.e. at some stage after E in the directed graph, and hence below it in any sequentialisation). We also have $C \in \mathrm{k}(M)$, and hence $N \ll M$, contradicting our choice.

5 Permutations of sequent rules

Since any sequent proof gives a proof net, the property of having the same net gives an equivalence on sequent proofs. In this section we investigate that equivalence, proving that two proofs give the same net iff each can be derived from the other by a sequence of innocuous transpositions of proof rules.

Specifically, we shall consider transpositions where the rules to be transposed are adjacent along a branch of a tree. Of course there is only a possibility of transposing such rules when the major formulae of the first are not hypotheses of the second. Having made that observation, in most instances the rules can be transposed. There is one exception: it is not always possible to move a switched rule up through an unswitched one.

The details of how the transposition is managed depend on the format of the rules involved, and the following examples provide the core of a definition by cases (unary, binary switched, binary unswitched, weakening) of what transposing rules means, and when it can be done. There are numerous (largely irrelevant) subcases depending on exactly which bits of a proof different formulae come from, and it would be tedious and unilluminating to provide the definition in full detail.

A unary rule can be transposed with anything, e.g. a (¬R) can be pushed up through an (∧R):

$$\dfrac{\dfrac{\dfrac{\cdots}{\Gamma, A \vdash B, \Delta} \quad \dfrac{\cdots}{\Gamma' \vdash C, \Delta'}}{\Gamma, \Gamma', A \vdash B \wedge C, \Delta, \Delta'}\,\wedge\mathrm{R}}{\Gamma, \Gamma' \vdash \neg A, B \wedge C, \Delta, \Delta'}\,\neg\mathrm{R}$$

can be transposed to (and yields the same net as)

$$\dfrac{\dfrac{\dfrac{\cdots}{\Gamma, A \vdash B, \Delta}}{\Gamma \vdash \neg A, B, \Delta}\,\neg\mathrm{R} \quad \dfrac{\cdots}{\Gamma' \vdash C, \Delta'}}{\Gamma \vdash \neg A, B \wedge C, \Delta}\,\wedge\mathrm{R}$$

This is reversible, and the unary rule can also be pushed back down again. Similarly unary rules can be transposed with switched rules and other unary rules.
Any two switched rules can be transposed. For example

$$\dfrac{\dfrac{\dfrac{\cdots}{\Gamma, A, B, C \vdash D, \Delta}}{\Gamma, A \wedge B, C \vdash D, \Delta}\,\wedge\mathrm{L}}{\Gamma, A \wedge B \vdash C \to D, \Delta}\,\to\mathrm{R}$$

transposes to, and yields the same net as

$$\dfrac{\dfrac{\dfrac{\cdots}{\Gamma, A, B, C \vdash D, \Delta}}{\Gamma, A, B \vdash C \to D, \Delta}\,\to\mathrm{R}}{\Gamma, A \wedge B \vdash C \to D, \Delta}\,\wedge\mathrm{L}$$

Two unswitched rules can also be permuted. For example

$$\dfrac{\dfrac{\dfrac{\cdots}{\Gamma, A \vdash C, \Delta} \quad \dfrac{\cdots}{\Gamma', B \vdash \Delta'}}{\Gamma, \Gamma', A \vee B \vdash C, \Delta, \Delta'}\,\vee\mathrm{L} \quad \dfrac{\cdots}{\Gamma'' \vdash D, \Delta''}}{\Gamma, \Gamma', \Gamma'', A \vee B \vdash C \wedge D, \Delta, \Delta', \Delta''}\,\wedge\mathrm{R}$$

transposes to

$$\dfrac{\dfrac{\dfrac{\cdots}{\Gamma, A \vdash C, \Delta} \quad \dfrac{\cdots}{\Gamma'' \vdash D, \Delta''}}{\Gamma, \Gamma'', A \vdash C \wedge D, \Delta, \Delta''}\,\wedge\mathrm{R} \quad \dfrac{\cdots}{\Gamma', B \vdash \Delta'}}{\Gamma, \Gamma', \Gamma'', A \vee B \vdash C \wedge D, \Delta, \Delta', \Delta''}\,\vee\mathrm{L}$$

A switched rule can always be moved below an unswitched one. For example:

$$\dfrac{\dfrac{\dfrac{\dots}{\Gamma \vdash A, C, D, \Delta}}{\Gamma \vdash A, C \vee D, \Delta}\,\vee\mathrm{R} \qquad \dfrac{\dots}{\Gamma' \vdash B, \Delta'}}{\Gamma, \Gamma' \vdash A \wedge B, C \vee D, \Delta, \Delta'}\,\wedge\mathrm{R}$$

can be transposed to

$$\dfrac{\dfrac{\dfrac{\dots}{\Gamma \vdash A, C, D, \Delta} \qquad \dfrac{\dots}{\Gamma' \vdash B, \Delta'}}{\Gamma \vdash A \wedge B, C, D, \Delta}\,\wedge\mathrm{R}}{\Gamma, \Gamma' \vdash A \wedge B, C \vee D, \Delta, \Delta'}\,\vee\mathrm{R}$$

There are however potential problems when we try to push a switched rule up through an unswitched one. The switched rule establishes two subproofs, and the unswitched rule can be moved freely only when its premises both come from the same subproof, as in the example above. However

$$\dfrac{\dfrac{\dfrac{\dots}{\Gamma \vdash A, C, \Delta} \qquad \dfrac{\dots}{\Gamma' \vdash B, D, \Delta'}}{\Gamma, \Gamma' \vdash A \wedge B, C, D, \Delta, \Delta'}\,\wedge\mathrm{R}}{\Gamma, \Gamma' \vdash A \wedge B, C \vee D, \Delta, \Delta'}\,\vee\mathrm{R}$$

cannot be transposed, since C and D come from separate subproofs. These considerations apply also to cut and contraction.

Finally we must consider weakening. Weakening can always be permuted, provided neither the formula introduced by the weakening, nor the formula that it is weakened against is active in the other rule.

The crucial lemma in showing that proofs which induce isomorphic proof nets are the same up to transposition of rules is the following.

Lemma 5. *Suppose sequent proofs P and P' induce the same proof net (up to isomorphism). Then there is a proof P'' of the same sequent, inducing the same proof net (up to isomorphism), such that P'' is obtained from P' by transposing independent rules, and P'' ends with the same application of the same rule as P. Furthermore, the subnets corresponding to the immediate subproofs of P are identical to those for P''.*

Proof. Since P and P' induce the same proof nets, any rule in P has a corresponding rule in P'. In particular this is the case for the bottom rule in P. If this is a rule with a single hypothesis (such as a switched rule for a binary connective), then the corresponding rule in P' can be permuted down without any difficulty. But if it has two hypotheses (such as an unswitched rule for a binary connective, or Cut), then there may be difficulties.

Consider the case where the final rule in P is

$$\dfrac{\Gamma_0 \vdash A, \Delta_0 \qquad \Gamma_1 \vdash B, \Delta_1}{\Gamma_0, \Gamma_1 \vdash A \wedge B, \Delta_0, \Delta_1}$$

(Other cases where we have a rule with two hypotheses are handled identically). Notice that $\mathrm{e}(A)$ and $\mathrm{e}(B)$ are disjoint, and that the proof net is $\mathrm{e}(A) \cup \mathrm{e}(B) \cup \{A \wedge B\}$. It follows that any rule in P but the last must come from one of the two empires, and in particular, if it has two active premisses, that either both are from $\mathrm{e}(A)$, or both from $\mathrm{e}(B)$. Since P and P' have the same proof net, this also applies to rules in P'.

Suppose the rule in P' corresponding to the final rule in P is

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}$$

and the rule below has active premisses C and D. If the rule is unswitched, then only one of these will be in $\Gamma, \Gamma', \Delta, \Delta'$, and we can permute without problems. However, if the rule is switched it appears possible that C and D might come from the different subproofs $\Gamma \vdash A, \Delta$ and $\Gamma' \vdash B, \Delta'$. Say, C comes from $\Gamma \vdash A, \Delta$, and D from $\Gamma' \vdash B, \Delta'$. But in that case C would come from $\mathrm{e}(A)$ and D from $\mathrm{e}(B)$, contradicting the discussion at the end of the last paragraph. It follows that both hypotheses come form the same subproof, and hence the rule can be permuted down.

Moreover, when the rule reaches the bottom of the proof, P', the sequents which form its hypotheses are determined as the doors of the two empires, and the subnets corresponding to the immediate subproofs are the two empires. This concludes the proof of the lemma.

Proposition 3. *Two sequent proofs P and P' induce isomorphic proof nets iff one can be obtained from the other by a sequence of transpositions of rules.*

Proof. The if direction is obvious. We prove the only if direction by induction over the tree structure of proof P, using lemma 5.

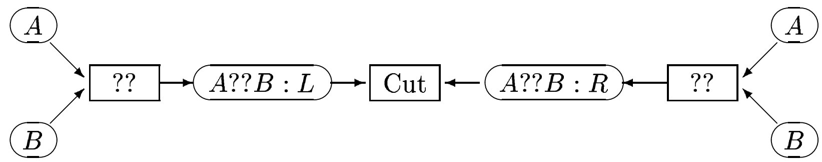
Thus proof nets exactly capture the notion of a local version of sequent proof, in which independent applications of rules can be carried out in either order.

This is not to say that these nets are definitive. They still make distinctions we might view as unnecessary. For example, contraction is binary, and different orderings are not identified (it is commutative, but not associative). Similarly there are questions about the handling of weakening. Handling these possible extra requirements is however beyond the immediate scope of this work.

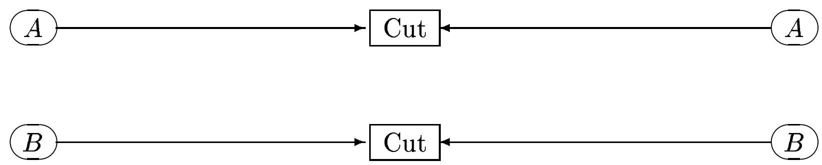
6 Dynamics of proof nets

We now turn our attention to a brief account of the dynamics of proof nets. Our goal is to mirror the reduction of classical sequent calculus. However we have just seen that nets identify different sequent proofs. Therefore the best we can hope for is that a sequence of reductions of nets corresponds to a sequence of reductions of proofs, interleaved with a sequence of restructurings by permuting independent rules. This is what we obtain if we establish that every proof reduction translates to a net reduction, and that if we have a net reduction, then there is some sequentialisation of the net, such that the sequent proof exhibits that reduction. These results will be given in detail in the companion paper [12]. Here we merely give a brief account of the dynamics.

We begin with the binary connectives. Notice that for each connective one rule is switched and the other is unswitched. This means that any logical cut on a binary connective looks like:

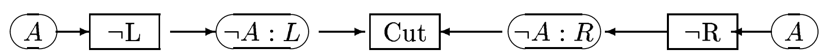


in which one of the rule nodes is switched, and the other unswitched. In order to reduce this cut we replace that segment of graph by

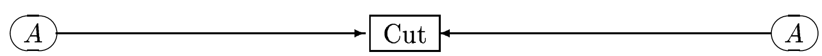


This local change both preserves and reflects the property of being a proof net, although it alters the topology of the Danos-Regnier graphs.

Similarly, a logical cut on the unary connective negation corresponds to a local change in the proof net:



becomes



Reductions of cuts through the structural rules of contraction and weakening are not, however, local changes. In the sequent calculus they involve copying or removing whole chunks of proof. The same is the case with proof nets. Moreover, they are non-deterministic: the chunk to be copied or deleted can be any subnet.

Consider first contraction. Suppose we cut against a proposition introduced by contraction, as in:

$$\dfrac{\dfrac{\dfrac{\mathcal{P}_1}{\Gamma \vdash A, A, \Delta}}{\Gamma \vdash A, \Delta}\,\mathrm{CR} \qquad \dfrac{\mathcal{P}_2}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

This reduces to

$$\cfrac{\cfrac{\cfrac{\mathcal{P}_1}{\Gamma \vdash A, A, \Delta} \quad \cfrac{\mathcal{P}_2}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\ \text{Cut} \qquad \cfrac{\mathcal{P}_2}{\Gamma', A \vdash \Delta'}}{\cfrac{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}{\cfrac{\vdots}{\Gamma, \Gamma' \vdash \Delta, \Delta'}}}\ \text{Cut}$$

The fragment of proof represented by vertical dots consists of contractions, contracting the corresponding elements of the two copies of Γ' and Δ'. This is mirrored in nets by a similar reduction. First we identify a subnet ending with the cut (i.e. we identify two disjoint subnets with the cut occurrences of A as doors). This sets up a correspondence between the net and the first proof. To reduce this we copy the subnet corresponding to $\mathcal{P}_2$, giving us two A nodes, and two copies of each of the doors. We contract the corresponding doors, remove the contraction node leading to A from the net corresponding to $\mathcal{P}_1$, and cut the two resulting occurrences of A against the A's coming from the two copies of $\mathcal{P}_2$.

There are two cases to consider for weakening. First the one in which we cut against the formula that has just been introduced by weakening:

$$\cfrac{\cfrac{\cfrac{\mathcal{P}_1}{\Gamma \vdash \Delta}}{\Gamma \vdash A, \Delta}\ \text{WR} \qquad \cfrac{\mathcal{P}_2}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Cut}$$

This reduces to

$$\cfrac{\cfrac{\mathcal{P}_1}{\Gamma \vdash \Delta}}{\cfrac{\vdots}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Cut}}$$

where the vertical dots are weakenings. In net terms we again identify subnets corresponding to $\mathcal{P}_1$ and $\mathcal{P}_2$. This time we delete $\mathcal{P}_2$, and add weakenings to $\mathcal{P}_1$ to preserve the doors of the combined proof.

The remaining case is one in which we have the following situation:

$$\cfrac{\cfrac{\cfrac{\mathcal{P}_1}{\Gamma \vdash A, \Delta}}{\Gamma \vdash A, B, \Delta}\ \text{WR} \qquad \cfrac{\mathcal{P}_2}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\ \text{Cut}$$

in which in the net, we have attached the formula B introduced by weakening to the formula A being cut. This reduces to

$$\cfrac{\cfrac{\cfrac{\mathcal{P}_1}{\Gamma \vdash A, \Delta} \quad \cfrac{\mathcal{P}_2}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Cut}}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\ \text{WR}$$

But in this proof we can no longer attach B to A, and we have to move the point of attachment of the weakening.

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