An Abstract Look at Realizability

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Abstract. This paper is about the combinatorial properties necessary for the construction of realizability models with certain type-theoretic properties. We take as our basic construction a form of tagging in which elements of sets are equipped with tags, and functions must operate constructively on tags. To complete the construction we allow a form of closure under quotients by equivalence relations. In this paper we analyse first the condition for a natural monoidal structure to be product structure, and then investigate necessary conditions for the realizability model to be locally cartesian closed and to have a subobject classifier.

Introduction

Realizability is a technique for constructing models in which all operations of a given type are computable, according to a given notion of computation. It extends the naive approach of enumerating elements and requiring that operations be computable with respect to the enumerations, in particular by allowing the construction of higher-order types. It produces extensional models which validate various forms of constructive reasoning, e.g. [10, 17, 19], and forms the basis for PER models of polymorphic lambda calculi e.g. [11]. All this work uses traditional intensional models of untyped computation, such as the Kleene algebra of partial recursive functions. However there is recent interest in extending this, for example to process models [1] or to the typed setting [13, 12].

These approaches tend to take quite a concrete approach, giving structures and building combinators into the definition. For example Longley's notion of typed pca assumes function spaces and application, and then uses them to construct a locally cartesian closed category (the category-theorists analogue of a type theory with dependent products). The purpose of the present paper is to attempt to reverse this. One of our results is that, modulo a condition to do with the way pairs are represented in the realizability model, if the realizability model is locally cartesian closed, then the model of computation has a weak form of function space, though not quite Longley's. This to some extent validates the use of combinatorial structures which have function spaces built in, and is typical of the form of our results. Broadly, they say that for the realizability model to support extensional forms of type structure, i.e. with both β

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and η laws, the underlying model of computation has to interpret corresponding combinators, but in a weak sense. This holds both for products and for function spaces. There is an exception to this pattern in the result which discusses what happens when the realizability model has a subobject classifier: in this case the model of computation must have a universal object, again in a weak sense, and thus that from the point of view of the model a typed form of realizability gives no extra generality over an untyped form.

We have chosen to use categorical technology and to couch our results in categorical terms. Thus, for us, a model of computation will be a category (for example the category with a single object, to be thought of as N, where the morphisms are partial recursive functions), and the existence of combinators will be given by structure on that category. There are two reasons for this choice. The first is that our account of the construction of a realizability model is essentially categorical. It is of course possible to give the construction in more set-theoretic language, and indeed this appears quite natural for the first part of the construction. However, set-theoretic constructions can be overly concrete. Our category-theoretic framework applies immediately to pointed cpo's, where there are at least two possible ways of assigning a set (include bottom or not). Moreover, if one uses a set-theoretic presentation, the second part of the construction (freely adjoining quotients of equivalence relations) is poorly motivated. It would not be clear why that particular definition should be chosen over a number of possible variants. Our second reason is that the categorical formulation gives a fairly clear idea of what the minimal supporting structure might be. Set-theoretic formulations have not.

In these senses the paper contrasts with recent work particularly by Longley [13] and Lietz and Streicher [12], in which the basis is taken as a typed generalisation of a partial combinatory algebra. We, like they, will be interested in when the construction yields a topos, and hence gives a full interpretation of higher-order logic. This is also a theme of Birkedal's work, see [2, 3], and his joint work in [4].

We present realizability toposes as the product of two constructions. First one takes a category (which corresponds to the typed partial combinatory algebra), and then one glues **Set** to it in a variant of the comma construction. This step is the categorical equivalent of forming a category in which objects are sets whose elements are tagged by possible realizers, e.g. natural numbers. The result should be a category with finite products, and we study the conditions under which it is so, or rather we study the conditions under which a natural monoidal structure gives finite products. In this event, it has long been known [6, 16] that in the examples derived from standard realizability the associated realizability topos is the exact completion, i.e. it is obtained essentially by freely adjoining quotients of equivalence relations. In the general case which we study, we do not necessarily get a cartesian closed category, still less a topos. We produce necessary conditions for *local* cartesian closure (dependent products, not just function spaces) and the existence of a subobject classifier (an object of truth values).

Our study of finite limits depends on the initial category being monoidal. This is the level of product structure exhibited by multiplicatives in linear logic. We show that finite products demand in addition combinators corresponding to diagonal and projections. These results can be read as saying that Birkedal was correct to use categories of partial maps as a basis for his theory, nothing significantly more general would have worked. In the case of function spaces, however, we get something slightly weaker than Birkedal's condition. Birkedal's condition is an analogue of the standard partial function space, in that every partial function is representable. Our results suggest that this is too strong for the current purpose, and all that is required is that some extension of any partial function be representable. This is a new notion of partial function space, which, to our knowledge, has not previously been encountered. Finally, we present our version of a result given independently by Birkedal and Lietz and Streicher, that if the realizability category has a subobject classifier, then the original category has a form of universal object. Our result is slightly more general than theirs, since it is independent of questions of cartesian closure inherent in their frameworks, and we give an explicit account of how it relates to untyped realizability.

The motivation for this work came from two directions. The first was to provide a general categorical account of traditional work on realizability. Our results show limitations on the use of typed forms of standard realizability in terms of the models they produce. There remains, however, modified realizability. It is possible to read the sets of "possible realizers" in modified realizability as a form of type, and hence to think of modified realizability as a form of typed realizability. Alas, our results show that this can not be if by typed realizability we mean either the construction given here, or, more particularly Longley's setting.

Our second motivation was to provide a case study giving the limitations of what could be achieved using these structures, but admitting the possibility of starting out with a very different model of computation, as in Abramsky's work on process realizability [1]. Here we believe that our results and techniques could be useful in narrowing down the design space.

A longer version of this paper is available from the authors. It contains a more substantial introduction as well as those proofs which have been excised for reasons of space.

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1 The $\mathcal{F}$-construction

There is a simple categorical generalisation of the construction of the category of partitioned assemblies, given by a variant of the standard comma construction.

We write **Ptl** for the category of sets and partial functions. The standard cartesian product of sets is no longer a categorical product, but it does provide a monoidal structure, which we shall use later.

Suppose $U : \mathbf{C} \longrightarrow \mathbf{Ptl}$ is a functor. Let $\mathcal{F}(\mathbf{C}, U)$ be the category whose objects are triples $(C, S, \sigma : S \longrightarrow U(C))$, where σ is total, and a map

$f : (C, S, \sigma) \longrightarrow (C', S', \sigma')$ is a (total) function $f : S \longrightarrow S'$ such that there exists a map $\phi : C \longrightarrow C'$ in $\mathbf{C}$ for which

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\sigma"'] & S' \arrow[d, "\sigma'"] \\ U(C) \arrow[r, "U(\phi)"] & U(C') \end{tikzcd}$$

commutes.

Notation. We shall always write $\mathcal{F}(\mathbf{C})$ leaving U understood. Instead of $(C, S, \sigma : S \longrightarrow U(C))$, we shall write a typical object of $\mathcal{F}(\mathbf{C})$ as $\sigma : S \longrightarrow U(C)$, using the fact that we can recover C from the notation $U(C)$. Finally, we shall write morphisms as pairs (f, ϕ). This is redundant in that the equality between morphisms is based only on the first component, but we shall need to use the second in some of our constructions.

We can think of this as a category of tagged sets. $\sigma : S \longrightarrow U(C)$ represents the tagging of the elements of S by realizers taken from $U(C)$. The functions are functions at the level of sets which can be traced by a function on tags.

Example 1. As part of the construction of a standard realizability topos, $\mathbf{C}$ can be taken to be the monoid of representable partial endo-functions on the partial combinatory algebra in use. In this case $\mathcal{F}(\mathbf{C})$ is the category of partitioned assemblies (the projective objects) in the associated realizability topos. In particular, if we take $\mathbf{C}$ to be the monoid of partial recursive functions on N, then we will get the projective objects of the classical effective topos.

The category $\mathcal{F}(\mathbf{C})$ always has equalizers, and we shall see that weak conditions on $\mathbf{C}$ ensure products in $\mathcal{F}(\mathbf{C})$. Similarly, weak conditions on $\mathbf{C}$ ensure that the exact completion $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ is locally cartesian closed.

Like a comma category, $\mathcal{F}(\mathbf{C})$ comes equipped with a number of functors.

Let $\mathbf{C}_{\mathrm{t}}$ be the inverse image along U of the subcategory of total functions, then there is a full functor $\mathsf{Y} : \mathbf{C}_{\mathrm{t}} \longrightarrow \mathcal{F}(\mathbf{C})$ defined by

$$[C \xrightarrow{f} C'] \longmapsto [(C, U(C), \mathrm{id}) \xrightarrow{(U(f), f)} (C', U(C'), \mathrm{id})]$$

or

$$[C \xrightarrow{f} C'] \longmapsto [(C \xrightarrow{\mathrm{id}} U(C)) \xrightarrow{(U(f), f)} (C' \xrightarrow{\mathrm{id}} U(C'))]$$

This becomes full and faithful when U is faithful.

Because of the existence condition in the definition of morphisms, there is no forgetful functor $\mathcal{F}(\mathbf{C}) \to \mathbf{C}$, however there is one $\mathcal{F}(\mathbf{C}) \to \mathbf{Set}$. More significantly, let C be an arbitrary object of $\mathbf{C}$, and $x : 1 \longrightarrow U(C)$ an arbitrary element of $U(C)$, then there is a full embedding $\nabla_{C,x} : \mathbf{Set} \longrightarrow \mathcal{F}(\mathbf{C})$ defined by

$$\nabla : [S \xrightarrow{f} S'] \longmapsto [(S \xrightarrow{x \circ !} U(C)) \xrightarrow{(f, U(\mathrm{id}))} (S' \xrightarrow{x \circ !} U(C))]$$

This definition is quite robust. If there are morphisms $\phi : C \longrightarrow C'$ and $\psi : C' \longrightarrow C$ such that $\phi x = x'$ and $\psi x' = x$, then $\nabla_{C,x}$ is naturally isomorphic to $\nabla_{C',x'}$.

Because of the existence condition in the definition of maps of $\mathcal{F}(\mathbf{C})$, it is clear that $\mathcal{F}(\mathbf{C})$ is equivalent to $\mathcal{F}(U[\mathbf{C}])$ where $U[\mathbf{C}]$ is the quotient of $\mathbf{C}$ with two maps identified when they have the same value under U (in other words, the category sitting in the middle of the (full and identity on objects)/faithful factorisation of U).

Notation. In order to make things less cluttered, from now on we shall write $\overline{(\ )}$ for the functor U, so $U(C) = \overline{C}$ and $U(\delta) = \overline{\delta}$.

2 Exact Completions

Our construction proceeds in two stages. We begin by constructing a base category using the $\mathcal{F}$-construction, and then we construct a better-behaved category from that using an exact completion. In other words our final category is a free exact category on a category obtained by means of the $\mathcal{F}$-construction.

Our results, then, rely on the fundamental property of an exact completion (cf. [7]): Given an exact category $\mathbf{A}$, let $\mathbf{P}$ be the full subcategory on the regular projectives of $\mathbf{A}$. Then $\mathbf{A}$ is an exact completion of a category with finite limits if and only if $\mathbf{P}$ is closed under finite limits and each object in $\mathbf{A}$ is covered by a regular projective (i.e. for every A in $\mathbf{A}$ there is a regular epi $P \longrightarrow\!\!\!\!\triangleright\, A$ from a regular projective). When this is the case, $\mathbf{A}$ is the exact completion of $\mathbf{P}$.

The crucial point here is that the base category of projectives, $\mathbf{P}$, which in our case is going to be $\mathcal{F}(\mathbf{C})$ must be left exact, and in the next section we explore conditions under which this is so.

3 Finite Limits

First, we observe that $\mathcal{F}(\mathbf{C})$ always has equalisers. This reduces the question to when $\mathcal{F}(\mathbf{C})$ has products. It is fairly easy to see when $\mathcal{F}(\mathbf{C})$ has a terminal object, though the condition seems both delicate and a little unnatural. However, characterising products seems more difficult.

Fortunately, in the cases we know about $\mathbf{C}$ can be taken to be a monoidal category, and $\overline{(\ )}$ a monoidal functor (cf. [9]). This means that $\mathcal{F}(\mathbf{C})$ has a candidate for a monoidal structure. The unit is given by $\psi : 1 \longrightarrow \overline{I}$, and the tensor by $(f : X \longrightarrow \overline{C}) \otimes (g : Y \longrightarrow \overline{D}) = \theta(f \times g) : X \times Y \longrightarrow \overline{C \otimes D}$, where ψ and $\theta : \overline{C} \times \overline{D} \longrightarrow \overline{C \otimes D}$ are the maps given by the monoidal structure of $\overline{(\ )}$. These definitions give valid objects of $\mathcal{F}(\mathbf{C})$ if and only if ψ and θ are total. In this case, the resulting structure is indeed monoidal. The verification is straightforward category theory, except that at some points we have to use the totality of various morphisms.

This allows us to ask a simpler question: when is this monoidal structure actually a product? This simplification is not without cost. We noted above that $\mathcal{F}(\mathbf{C})$ is equivalent to $\mathcal{F}(\overline{\mathbf{C}})$ where $\overline{\mathbf{C}}$ is the quotient of $\mathbf{C}$ with two maps identified when they have the same value under $\overline{(\ )}$. This suggests that without loss of generality we can take $\overline{(\ )}$ to be faithful. This is not, unfortunately, the case. The problem is that the monoidal structure on $\mathbf{C}$ does not necessarily transfer to one on $\overline{\mathbf{C}}$. The reason is that unless θ is iso, the monoidal tensor does not necessarily respect equivalence of maps. But, unless $\overline{(\ )}$ is faithful we cannot completely reflect properties of $\mathcal{F}(\mathbf{C})$ back into properties of $\mathbf{C}$. This explains why in general we prove properties up to the functor $\overline{(\ )}$, leaving the cleaner and perhaps more interesting case where $\overline{(\ )}$ is faithful to corollaries. This is first evident in the characterisation of when the monoidal unit on $\mathcal{F}(\mathbf{C})$ is terminal.

Lemma 1. *$\psi : 1 \longrightarrow \overline{I}$ is terminal in $\mathcal{F}(\mathbf{C})$ if and only if for each object C of $\mathbf{C}$ there is a map $t_C : C \longrightarrow I$ such that $\overline{t_C} = \psi \circ !$.*

$$\begin{tikzcd}
\overline{C} \arrow[r, "!"] \arrow[d, "\mathrm{id}"'] & 1 \arrow[d, "\psi"] \\
\overline{C} \arrow[r, "\overline{t_C}"] & \overline{I}
\end{tikzcd}$$

Proof. If $\psi : 1 \longrightarrow \overline{I}$ is terminal, then we obtain t_C by considering the terminal map from $\mathrm{id} : \overline{C} \longrightarrow \overline{C}$ to ψ (the diagram is as above). Conversely, given such a family of maps, ψ is weakly terminal because for any $f : X \longrightarrow \overline{C}$, we have the following diagram (note that the upper triangle commutes because f is total).

$$\begin{tikzcd}
X \arrow[r, "!"] \arrow[d, "f"'] & 1 \arrow[d, "\psi"] \\
\overline{C} \arrow[r, "\overline{t_C}"] \arrow[ur, "!"] & \overline{I}
\end{tikzcd}$$

However, maps into ψ are unique, when they exist, because maps into 1 are. This establishes that ψ is terminal. $\square$

Now, if the unit of a monoidal category is terminal, then there are candidates for left and right projections from the tensor:

$$\pi_{0,XY} = \rho_X(\mathrm{id}_X \otimes t_Y) : X \otimes Y \longrightarrow X \qquad \pi_{1,XY} = \lambda_Y(t_X \otimes \mathrm{id}_Y) : X \otimes Y \longrightarrow Y$$

This allows us to ask the question of when the monoidal tensor is a product, in the precise sense that these projections together form a product cone.

Lemma 2. *In the case that $\psi : 1 \longrightarrow \overline{I}$ is terminal in $\mathcal{F}(\mathbf{C})$, then the candidates for projections above form product cones if and only if for each object c of $\mathbf{C}$ there is a map $d_C : C \longrightarrow C \otimes C$ such that $\overline{d_C} = \theta \circ \Delta_{\overline{C}}$, where $\Delta_{\overline{C}} : \overline{C} \longrightarrow \overline{C} \times \overline{C}$ is the ordinary cartesian diagonal.*

Proof. (Sketch) First, suppose that the tensor is cartesian product. Then the tensor of $\mathrm{id} : \overline{C} \longrightarrow \overline{C}$ with itself is $\theta : \overline{C} \times \overline{C} \longrightarrow \overline{C \otimes C}$. This must have a diagonal

$$\begin{tikzcd}
\overline{C} \arrow[r, "D"] \arrow[d, "\mathrm{id}"'] & \overline{C} \times \overline{C} \arrow[d, "\theta"] \\
\overline{C} \arrow[r, "\overline{d_C}"] & \overline{C \otimes C}
\end{tikzcd}$$

Composing with the projections we see that D must be the diagonal $\Delta_{\overline{C}}$, and the square then yields $\overline{d_C} = \theta \circ \Delta_{\overline{C}}$, as required.

For the converse, suppose we have two maps

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d] & X \arrow[d, "f"] \\
\overline{C} \arrow[r, "\overline{a}"] & \overline{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "k"] \arrow[d] & Y \arrow[d, "g"] \\
\overline{C} \arrow[r, "\overline{b}"] & \overline{B}
\end{tikzcd}$$

then we can form the pairing

$$\begin{tikzcd}
Z \arrow[r, "\Delta"] \arrow[d] & Z \times Z \arrow[r, "h \times k"] \arrow[d] & X \times Y \arrow[d, "f \times g"] \\
\overline{C} \arrow[r, "\Delta"] \arrow[d] & \overline{C} \times \overline{C} \arrow[r, "\overline{a} \times \overline{b}"] \arrow[d, "\theta"'] & \overline{A} \times \overline{B} \arrow[d, "\theta"] \\
\overline{C} \arrow[r, "\overline{d_C}"] & \overline{C \otimes C} \arrow[r, "\overline{a \otimes b}"] & \overline{A \otimes B}
\end{tikzcd}$$

given by composing the obvious "diagonal" on $Z \longrightarrow \overline{C}$ (the left-hand half of the diagram) with the tensor product. □

Note that Δ and θ are natural considered as transformations between functors $\mathbf{C} \longrightarrow \mathbf{Ptl}$, hence $\overline{d_C}$ is natural in C. Thus, if $\overline{(\ )}$ is faithful, then d_C itself is natural in C. However, although ψ is natural in C, $\underline{!}$ is only natural in the subcategory of total maps. Whence $\overline{t_C}$ (and hence t_C, if $\overline{(\ )}$ is faithful) is natural only in the category of total maps in $\mathbf{C}$.

Moreover, it is not necessarily the case that $\overline{I}$ is isomorphic to 1, or that $\overline{X \otimes Y}$ is isomorphic to $\overline{X} \times \overline{Y}$. However:

Lemma 3. *There is a map* $e : X \otimes Y \longrightarrow X \otimes Y$ *such that* $\overline{e}$ *is an idempotent split by*

$$\overline{X \otimes Y} \xrightarrow{(\overline{\pi_0}, \overline{\pi_1})} \overline{X} \times \overline{Y} \xrightarrow{\theta} \overline{X \otimes Y}$$

where $\pi_0 = \rho(\mathrm{id} \otimes t)$ *and* $\pi_1 = \lambda(t \otimes \mathrm{id})$.

Similarly t_I is an endomorphism on I, such that $\overline{t_I}$ is split by

$$\overline{I} \xrightarrow{\ !\ } 1 \xrightarrow{\ \psi\ } \overline{I}$$

In summary:

Lemma 4. *If* $\mathbf{C}$ *is a symmetric monoidal category and* $\overline{(\ )}$ *a faithful symmetric monoidal functor* $\mathbf{C} \longrightarrow \mathbf{Ptl}$, *for which the structural maps* $\psi : 1 \longrightarrow \overline{I}$ *and* $\theta_{C,D} : \overline{C} \times \overline{D} \longrightarrow \overline{C \otimes D}$ *are total, then* $\mathcal{F}(\mathbf{C})$ *carries a symmetric monoidal structure. This is a product structure, i.e. the unit is terminal, and the monoidal product together with projections defined from terminal maps and monoidal structure forms product cones, if and only if for each object* C *of* $\mathbf{C}$ *there are maps* $t_C : C \longrightarrow I$ *and* $d_C : C \longrightarrow C \otimes C$ *such that* $\overline{t_C} = \psi!$ *and* $\overline{d_C} = \theta_{C,C}\Delta_{\overline{C}}$. *In addition,* $\overline{d_C}$ *is natural in* C *(though* $\overline{t_C}$ *is not).*

If $\mathcal{F}(\mathbf{C})$ is left exact, then we can take its exact completion. This is our candidate for a topos. In the next two sections we see what we can say about $\mathbf{C}$ when this category is (locally) cartesian closed or has a subobject classifier. It is simpler to deal with the subobject classifier first.

4 Subobject Classifiers and Universal Objects

From this point we shall make the following running assumptions: $\mathbf{C}$ is a symmetric monoidal category and $\overline{(\ )}$ a symmetric monoidal functor $\mathbf{C} \longrightarrow \mathbf{Ptl}$, for which the structural maps $\psi : 1 \longrightarrow \overline{I}$ and $\theta_{C,D} : \overline{C} \times \overline{D} \longrightarrow \overline{C \otimes D}$ are total. Moreover we require the existence of families of maps $t_C : C \longrightarrow I$ and $d_C : C \longrightarrow C \otimes C$ such that $\overline{t_C} = \psi \circ !$ and $\overline{d_C} = \theta_{C,C} \circ \Delta_{\overline{C}}$, as in the last section. These assumptions ensure that $\mathcal{F}(\mathbf{C})$ is left exact, with cartesian structure derived from the monoidal structure of $\mathbf{C}$.

In this section we investigate the connection between the existence of a subobject classifier in $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ and universal objects in $\mathbf{C}$. As before, our main result takes its cleanest form when U is faithful, but can be deduced immediately from a more technical statement which holds in general.

Definition 1. *The category* $\mathbf{C}$ *has a* universal object W *if each object* C *of* $\mathbf{C}$ *is a retract of* W.

Proposition 1. *If the category* $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ *has a subobject classifier, then there is an object* W *of* $\mathbf{C}$, *such that for each object* C *of* $\mathbf{C}$ *there are morphisms* $\gamma : C \longrightarrow W$ *and* $\delta : W \longrightarrow C$ *such that* $\overline{\delta\gamma}$ *is the identity on* $\overline{C}$.

Intuitively, modulo U, W is universal in $\mathbf{C}$. If U is faithful, then this immediately implies that $\delta\gamma = \mathrm{id}_C$.

Corollary 1. *If the category $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ has a subobject classifier, and the functor U is faithful, then the category $\mathbf{C}$ has a universal object.*

This result is closely connected to one in [12, 2] obtained for the subcategory of an exact completion as above which is the regular completion of $\mathbf{P}$, see [5].

Our proof builds on previous analysis of subobject classifiers in exact completions. The following is a slight variant of Menni [14].

Definition 2. *A map $u : W \longrightarrow V$ is a (weakly) weak proof classifier if every map in the category appears as weakly equivalent to a (weak) pullback of u: i.e. for every map $a : X \longrightarrow A$ there is a diagram*

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[dr, "a"'] & X' \arrow[l, shift left, "k"] \arrow[r] \arrow[d, "a'"] & W \arrow[d, "u"] \\ & A \arrow[r, "f"] & V \end{tikzcd} \tag{1}$$

where the square is a (weak) pullback, and the triangles commute.

In an exact category $\mathbf{A}$ where every object is covered by a regular projective, a weakly weak proof classifier is what can be traced directly in the full subcategory $\mathbf{P}$ of projectives when $\mathbf{A}$ has a subobject classifier. If in addition $\mathbf{P}$ is a left exact subcategory (as when $\mathbf{A}$ is its exact completion), then any weakly weak proof classifier is actually a weak proof classifier in the sense of Menni (the weak pullback in the definition can always be taken to be a pullback).

We will also need a further technical result, establishing a factorisation property which generalises a standard lemma for subobject classifiers, and is best seen in the abstract:

Lemma 5. *Suppose that $u : W \longrightarrow V$ is a (weakly) weak proof classifier, and that $a : X \longrightarrow A$ is an arbitrary map. The (weakly) weak proof classifier produces a diagram*

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[dr, "a"'] & X' \arrow[l, shift left, "k"] \arrow[r] \arrow[d, "a'"] & W \arrow[d, "u"] \\ & A \arrow[r, "f"] & V \end{tikzcd}$$

Suppose, now that $b : Y \longrightarrow A$ makes f true, in the sense that $f \circ b$ factors through u, then b factors through a.

Proof (Proposition 1, sketch). Taking a weak proof classifier

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "w"'] & P \arrow[d, "v"] \\ \overline{W} \arrow[r, harpoon, "\overline{\phi}"] & \overline{V} \end{tikzcd} \tag{2}$$

we prove that W is "universal" (quotation marks indicate that this holds modulo $\overline{(\ )}$).

To prove that C is a "retract", classify

$$\begin{tikzcd} \overline{C} \arrow[r, "\mathrm{id}_{\overline{C}}"] \arrow[d, "\overline{\mathrm{id}_C}"'] & \overline{C} \arrow[d, "\overline{t_C}"] \\ \overline{C} \arrow[r, "\overline{t_C}"] & \overline{I} \end{tikzcd} \tag{3}$$

giving a map $\overline{\gamma} : \overline{C} \longrightarrow \overline{W}$. This is total, and we use it as an object of $\mathcal{F}(\mathbf{C})$ in order to establish the existence of a "retraction". $\square$

5 Function Spaces

In this section we deal with conditions for the cartesian closure of $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$. We continue with the running assumption made at the start of section 4: that $\mathbf{C}$ is a symmetric monoidal category, and $U : \mathbf{C} \longrightarrow \mathbf{Ptl}$ a symmetric monoidal functor satisfying certain conditions so that $\mathcal{F}(\mathbf{C})$ is a left exact category with product structure constructed from the monoidal structure on $\mathbf{C}$. We shall abuse the structure and refer to a map f in $\mathbf{C}$ as *total* just when its image under U, $\overline{f}$, is total.

As in section 4, our work builds heavily on previous work on properties of exact completions. One of the major lessons of [8] is that in this context it is easier to deal with local cartesian closure, than simple cartesian closure. So it is an important fact that exact completion is a local construction.

Lemma 6. *Let $\mathbf{P}$ be a left exact category with exact completion $\mathbf{A}$. Then for any object P of $\mathbf{P}$, the slice $\mathbf{A}/P$ is the exact completion of $\mathbf{P}/P$.*

We shall use this in combination with the following facts about cartesian closure of exact completions.

Lemma 7. *Let $\mathbf{P}$ be a left exact category with exact completion $\mathbf{A}$. If $\mathbf{A}$ is cartesian closed, then for any objects P and Q of $\mathbf{P}$, there is a weak evaluation $\epsilon : F \times P \longrightarrow Q$ from P to Q in $\mathbf{P}$, i.e. any map $\phi : X \times P \longrightarrow Q$ can be expressed as $\epsilon \circ (f \times \mathrm{id}_P)$ for some $f : X \longrightarrow F$ (here all of the last part of the statement takes place in $\mathbf{P}$).*

We shall need to deal with a monoidal structure that approximates a product. First some notation.

Notation. The structure on $\mathbf{C}$ gives operations which we can loosely think of as pairing and projections, and which we shall write:

$$\langle f, g\rangle = (f \otimes g) \circ d_Z : Z \longrightarrow X \otimes Y$$

$$\pi_0 = \rho_X \circ (\mathrm{id}_X \otimes t_Y) : X \otimes Y \longrightarrow X \qquad \pi_1 = \lambda_Y \circ (t_X \otimes \mathrm{id}_Y) : X \otimes Y \longrightarrow Y$$

We shall use (a, b), p_0 and p_1 for the usual pairing and projections from a categorical product.

Returning to our main interest, if

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\sigma"'] & Y \arrow[d, "\sigma'"] \\ \overline{C} \arrow[r, harpoon, "\overline{g}"] & \overline{D} \end{tikzcd}$$

a morphism in $\mathcal{F}(\mathbf{C})$, then we can replace g by any morphism f such that $\overline{f}$ extends $\overline{g}$. It follows that if we have constructed some morphism f in $\mathbf{C}$, then we will not be able to prove that $\overline{f}$ is a particular *partial* function h, only that it is an extension of h. Moreover, we have seen that our monoidal structure is not a product, but the product is related by retraction. This motivates the following definition:

Definition 3. *Suppose* $\mathbf{C}$ *is a monoidal category equipped with a functor* $\overline{(\ )}$ *into* **Ptl**, *together with families of maps* $t_C : C \longrightarrow I$ *and* $d_C : C \longrightarrow C \otimes C$, *as in our standard structure. Then we say that a morphism* $f : A \longrightarrow B$ extends *a morphism* $g : A \longrightarrow B$ *(*$g \subseteq f : A \longrightarrow B$*) if* $\overline{g} \subseteq \overline{f}$. *We now say that a map* $\epsilon : F \otimes C \longrightarrow C'$ *is a weak partial evaluation from* C *to* C' *if for every map* $\phi : X \otimes C \longrightarrow C'$ *there is a total* $f : X \longrightarrow F$, *such that* $\epsilon \circ (f \otimes \mathrm{id}_C) \circ e$ *extends* $\phi \circ e$, *where* $e : X \otimes C \longrightarrow X \otimes C$ *is the "*η*-retraction" for pairing* $e = \langle \pi_0, \pi_1 \rangle$.

This differs from a standard definition of partial function space in that it does not demand that arbitrary partial functions be represented, only that some extension of them be, and also in that the equation unexpectedly passes through the "η-retraction" for pairing. This can be viewed as saying that the equation does not have to hold on the whole of $X \otimes C$, but only on those elements which are actually ordered pairs.

Moreover, the definition we have given depends upon U to give notions of totality and extension for morphisms in $\mathbf{C}$. However, instead of deriving these notions directly from U, we could instead use the "diagonal" and "terminal" maps in $\mathbf{C}$ to give internal definitions. This is a standard trick in p-categories, and fortunately agrees with our other definition. It follows that if we regard the "diagonal" and "terminal" maps as part of our structure, then we can reasonably suppress mention of this dependence on U.

Proposition 2. *Suppose* $\mathbf{C}$ *and* $U : \mathbf{C} \longrightarrow \mathbf{Ptl}$ *satisfy our running assumptions, then if the exact completion of* $\mathcal{F}(\mathbf{C})$ *is locally cartesian closed, then for any pair of objects* C *and* C' *of* $\mathbf{C}$*, there is an object* F *of* $\mathbf{C}$ *and a map* $\epsilon : F \otimes C \longrightarrow C'$*, such that for any map* $\phi : X \otimes C \longrightarrow C'$*, there is a map* $f : X \longrightarrow F$ *such that* $\overline{f}$ *is total and* $\overline{\epsilon \circ (f \otimes \mathrm{id}_C) \circ e}$ *extends* $\overline{\phi \circ e} : X \otimes C \longrightarrow C'$*, where* $e : X \otimes C \longrightarrow X \otimes C$ *is the* η*-retraction for pairing* $e = \langle \pi_0, \pi_1 \rangle$*, as before.*

Corollary 2. *If in the above the functor* U *is faithful, then the exact completion of* $\mathcal{F}(\mathbf{C})$ *is locally cartesian closed if and only if* $\mathbf{C}$ *has weak partial evaluations.*

The corollary follows immediately from the proposition, which, however, is technically the most demanding result in the paper. The proof depends on the use of cartesian closure in a slice category to define the weak partial evaluation. More specifically we work in a slice over a set derived from the possible subfunctions of the identity on C in order to get a generic function space. Details are in the full version of this paper.

6 Consequences for Realizability

In this section we draw out the consequences of our previous results in the case that most interests us. We shall suppose that $\mathbf{C}$ is a category of sets and functions and that $\overline{(\;)}$ is the underlying set functor. Thus $\overline{(\;)}$ is faithful. What we have in mind is that $\mathbf{C}$ is the category obtained from some form of typed partial applicative structure, as in Longley [13], but part of the game is to see how much of that structure we can reconstruct from properties of the resulting realizability category.

In section 3, we examined the case when a monoidal structure induced a product on $\mathcal{F}(\mathbf{C})$. In lemma 4 we showed that in this case we had a diagonal $d_C : C \longrightarrow C \otimes C$ and collection of maps into the unit $t_C : C \longrightarrow I$, satisfying certain properties. We have seen that these induce projections $\pi_0 : X \otimes Y \longrightarrow X$ and $\pi_1 : X \otimes Y \longrightarrow Y$, and a form of pairing $\langle a, b \rangle : z \longrightarrow X \otimes Y$. This pairing satisfies the beta laws:

$$\pi_0 \circ \langle a, b \rangle = a \quad \text{and} \quad \pi_1 \circ \langle a, b \rangle = b$$

but not necessarily the eta law

$$\langle \pi_0, \pi_1 \rangle = \mathrm{id} : X \otimes Y \longrightarrow X \otimes Y$$

The result is that we have something which is almost, but not quite, a category of partial maps on a category with finite products. It is interesting to compare with the formalisms given in [15], and to check when the equations listed there are satisfied. It turns out that the transformations have the correct naturality properties, but equations whose domains are tensors $X \otimes Y$ are valid only when composed with the retraction on $X \otimes Y$. We can therefore obtain a category of partial maps by splitting suitable idempotents. Since idempotents split in **Ptl**, $\overline{(\ )}$ extends to the resulting category (though it is not obviously still faithful). Now

Lemma 8. *If in* $\mathbf{C}$, C *is a retract of* D

$$\begin{tikzcd} C \arrow[r, shift right, "i"'] & D \arrow[l, shift right, "r"'] \end{tikzcd}$$

then an object $f : X \longrightarrow \overline{C}$ *of* $\mathcal{F}(\mathbf{C})$ *is isomorphic to* $\bar{\imath} \circ f : X \longrightarrow \overline{D}$.

Corollary 3. *If* $\overline{(\ )} : \mathbf{C} \longrightarrow \mathbf{Ptl}$ *and* $\mathbf{D}$ *is a category obtained from* C *by splitting idempotents, then* $\overline{(\ )}$ *extends to* $\mathbf{D}$*, and* $\mathcal{F}(\mathbf{D})$ *is equivalent to* $\mathcal{F}(\mathbf{C})$.

So this process does not affect the resulting category.

This means that if $\mathcal{F}(\mathbf{C})$ is a left exact category (or more exactly if it is lex and that structure is obtained from monoidal structure on $\mathbf{C}$), then $\mathbf{C}$ must already have interpretations of the combinators for pairing and unpairing satisfying similar properties to the pairing and unpairing in **Ptl**. At this level, then we parallel very closely the structure used by Birkedal [3], with only the minor details of certain equations holding only up to η.

Suppose now, that $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ is locally cartesian closed. Then by corollary 2, $\mathbf{C}$ has weak partial evaluations. This means that for any pair C, D of objects of $\mathbf{C}$, there is an object which we can call $[C \rightharpoonup D]$ together with an evaluation map $\epsilon : [C \rightharpoonup D] \otimes C \longrightarrow D$. This generates an "application" in **Ptl**: $\overline{\epsilon} \circ \theta : \overline{[C \rightharpoonup D]} \times \overline{C} \longrightarrow \overline{D}$. This is more general than the structure used by Birkedal. We use it to construct a partial combinatory type structure in the sense of Longley [13].

The type world T is the set $\mathbf{C}_0$ of objects of $\mathbf{C}$, the binary product operation $C \times D$ is tensor product $C \otimes D$, and the arrow type is given by the weak partial evaluations $[C \rightharpoonup D]$. The associated family of sets is $(A_C | C \in \mathbf{C}_0) = (\overline{C} | C \in \mathbf{C}_0)$, and the application functions $\overline{[C \rightharpoonup D]} \times \overline{C} \longrightarrow \overline{D}$ are as above.

Longley's structure also requires s and k combinators, along with combinators for pairing and first and second projections. These are obtained by currying corresponding maps in $\mathbf{C}$. For example, the combinator $k \in \overline{[C \rightharpoonup D \rightharpoonup C]}$ is obtained from $\pi_0 : C \otimes D \longrightarrow C$. We first curry to get a map $k_1 : C \longrightarrow [D \rightharpoonup C]$, and then again to get $k_2 : I \longrightarrow [C \rightharpoonup D \rightharpoonup C]$, apply $\overline{(\ )}$ to get a (total) function $\overline{I} \longrightarrow \overline{[C \rightharpoonup D \rightharpoonup C]}$, and finally compose this with $\psi : 1 \longrightarrow \overline{I}$ to get k. The construction of s is similar, this time starting with

$$\begin{aligned} (C \rightharpoonup D \rightharpoonup E) \otimes (C \rightharpoonup D) \otimes C &\xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes d} (C \rightharpoonup D \rightharpoonup E) \otimes (C \rightharpoonup D) \otimes C \otimes C \\ &\xrightarrow{\sim} (C \rightharpoonup D \rightharpoonup E) \otimes C \otimes (C \rightharpoonup D) \otimes C \end{aligned}$$

$$\xrightarrow{\epsilon\otimes\epsilon} \quad (D \rightharpoonup E)\otimes D$$
$$\xrightarrow{\epsilon} \quad E$$

Pairing and unpairing combinators are also obtained in this way, and satisfy the requisite equations. This can be seen from the following lemma.

Lemma 9. *Suppose $f : C \longrightarrow D$ is curried to give $F : I \longrightarrow [C \rightharpoonup D]$, then for all $c \in \overline{C}$, if $\overline{f}(c)$ is defined, then so is $(\overline{\epsilon} \circ \theta \circ (\overline{F} \times \mathrm{id}_{\overline{C}})(\psi, c)$, and they are equal.*

Proof. $(\overline{\epsilon} \circ \theta \circ (\overline{F} \times \mathrm{id}_{\overline{C}})(\psi, c) = (\overline{\epsilon \circ (F \otimes \mathrm{id}_C)} \circ \theta)(\psi, c)$, and the result follows from corollary 2. □

We thus have a partial combinatory type structure. This in turn generates a graph $\mathbf{C}'$ equipped with a graph morphism into **Ptl**: the vertices are the same as the objects of $\mathbf{C}$, and the edges are the partial functions induced from the arrow types by application. It is irritating that $\mathbf{C}'$ is not necessarily a category, but it may fail to be closed under composition (we only know that the composite of two partial functions in the graph can be extended to a third). But we only need this lax structure, not a full category, to define $\mathcal{F}(\mathbf{C}')$. Since any partial function obtained from $\mathbf{C}$ is extended by a partial function obtained from $\mathbf{C}'$, there is an embedding $\mathcal{F}(\mathbf{C}) \longrightarrow \mathcal{F}(\mathbf{C}')$. Unfortunately, this is not necessarily an equivalence. The problem is that there is no reason why a partial function obtained from $\mathbf{C}'$ should extend to one obtained from $\mathbf{C}$. One way of viewing the problem is that in a typed pca, as a consequence of the K combinator, every element of a type is named by a constant function. This is not the case for us. The K combinator corresponds to a projection. Suppose, however, that $\mathbf{C}$ is concrete in the sense that every element of $\overline{C}$ is obtained from a morphism $I \longrightarrow C$. In this case every partial function in $\mathbf{C}'$ is already in $\mathbf{C}$, and the realizability structure obtained categorically is identical to that obtained from the partial combinatory type structure.

We now turn our attention to the case when $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ has a subobject classifier. In that case we have that $\mathbf{C}$ has a universal object V, and applying lemma 8, we get that $\mathcal{F}(\mathbf{C})$ is equivalent to $\mathcal{F}(\mathbf{M})$ where $\mathbf{M}$ is the monoid of endomorphisms on V.

Corollary 4. *If $\mathcal{F}(\mathbf{C})$ is a topos, then it is equivalent to the topos constructed using the monoid of endomorphisms of the universal object in $\mathbf{C}$.*

Putting these observations together, we can see that if $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ is a topos, then, much as in Scott [18], $\overline{V}$ is a partial combinatory algebra, and if $\mathbf{C}$ is concrete, then the topos obtained is the conventional realizability topos from this algebra.

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