An abstract look at realizability

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This paper is about purely categorical approaches to realizability, and contrasts with recent work particularly by Longley [14] and Lietz and Streicher [13], in which the basis is taken as a typed generalisation of a partial combinatory algebra. We, like they, will be interested in when the construction yields a topos, and hence gives a full interpretation of higher-order logic. This is also a theme of Birkedal's work, see [1, 2], and his joint work in [3]. Birkedal makes considerable use of the construction we study.

We present realizability toposes as the product of two constructions. First one takes a category (which corresponds to the typed partial combinatory algebra), and then one glues **Set** to it in a variant of the comma construction. This, as we shall see, has the effect of improving the categorical properties of the algebra category. Then one takes an exact completion of the result. This also has the effect of improving the categorical properties. Formally the main result of the paper is that the result is a topos just (modulo some technical conditions) when the original category has a universal object.

Early work on realizability (e.g.[12, 22], or see [23]) is characterised by its largely syntactic nature. The core definition is when a sentence of some formal logic is realised, and the main interest is in when certain deductive principles (such as Markov's rule) are validated. Martin Hyland's invention

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of realizability toposes [10] advances on this, not only in the simplicity of the construction, but by providing a semantic framework in which the formal logics can naturally be interpreted.

Hyland was strongly motivated in his work by a then recent approach to the Boolean-Valued models used in Set Theory, originally introduced in an obscure paper by Higgs [9], but subsequently exposed much more widely in substantial articles by Fourman and Scott [8] and Scott [20]. Suppose one has a complete Boolean algebra B. Then a B-valued predicate on a set X can simply be taken to be a function $P\colon X \longrightarrow B$. If we have a binary predicate on X and Y, this can be taken to be a function $Q\colon X \times Y \longrightarrow B$.

The propositional operators can be interpreted using the boolean algebra structure, and quantifiers can be interpreted using the completeness of the algebra, so that

$$\forall y.\, Q(x,y) = \lambda x.\, \bigwedge_{y \in Y} .\, Q(x,y)\colon X \to B$$

The conventional approach to Boolean-valued models uses a transfinite induction. An ordinally indexed hierarchy (V_α) of sets of rank α is set up. $V_{\alpha+1}$ consists of functions $V_\alpha \longrightarrow B$, with inductively defined relations of equality and membership. This gives a model of Zermelo-Fraenkel set theory, and hence a topos.

Higgs' approach is to define a category in which the objects are sets equipped with a B-valued symmetric and transitive binary predicate, in other words a partial equivalence relation. Morphisms are defined naturally as B-valued maps of partial equivalence relations. The resulting category is easily seen to be equivalent to that obtained from the standard transfinite construction. The major difference is that there is no satisfactory global interpretation of membership. Note that the relations used are not necessarily reflexive (i.e. are not quite models of the theory of equality). This corresponds to the presence of the B-valued predicates, which produce by comprehension sets in which elements exist (i.e. are equal to themselves) only to the extent of some arbitrary element of B.

Hyland's insight [10] was that one could simply replace the Boolean valued interpretation of first-order logic in Higgs' construction by a realizability interpretation, and still produce a topos. This produced a setting in which syntactic realizability constructions could be seen as arising naturally out of the interpretation of higher-order structures in the topos.

3 A natural problem then was to combine Higgs' B-valued set construction, and Hyland's realizability one in a single formal framework. What they have in common is an interpretation of propositional first-order logic over **Set** and then the construction of the category of partial equivalence relations. It is natural to ask for sufficient conditions on the logic for the construction to yield a topos. This is the approach taken in the notion of tripos [11], and developed by Pitts in his thesis [16], cf. [17]. However, by its very nature, the theory of triposes is more general than the narrow study of realizability. In a rough analogy to our picture, the tripos construction can also be split into two phases. First construct the category whose objects are the interpretations of the non-standard predicates over **Set** given in the logic (this is not quite the associated Grothendieck category, since that forces undesirable distinctions between subsets of different objects). Then construct the final category by quotienting equivalence relations, see [1].

The study of the effective topos was given a push by Moggi's realization that it provided a natural place to see a particular model of the second-order lambda calculus (the PER model). This led to a spate of work, and indirectly to Peter Freyd asking the question of which universal property characterized the effective topos. Answers to this were given in [5], [19] and [4]. Broadly speaking, these characterise the topos as the free exact category on some other structure. There are two different choices for this substructure, either a left exact or a regular category. These two choices single out two subcategories of the topos: the partitioned assemblies and the assemblies. The topos is the free exact category on the regular category of assemblies, and the free exact category on the left exact category of partitioned assemblies. It turns out that the assemblies are also the free regular category on the left exact category of partitioned assemblies.

The two categories can be characterised internally in the topos. The partitioned assemblies are the regular projectives and the assemblies are the $\neg\neg$-separated objects. They are quite different in flavour. As the full subcategory of separated objects for a topology in the topos, the assemblies have strong categorical properties: apart from being a regular category and they are locally cartesian closed. Although the projectives are closed under finite limits, they have nothing like this categorical structure. They are, however, simpler to construct purely categorically. They can be constructed by a simple variant of the comma construction. The assemblies require a construction involving relations.

There is a further interesting difference between them. The assemblies are exactly what one gets from the middle stage of the general tripos construction sketched out above. But it does not seem to be similarly possible to construct an analogue of the category of projectives. This suggests to us that this category of projectives might have more of an intrinsic connection with realizability, and it is this construction that we shall study in detail.

Specifically, we shall start with a category equipped with a functor into the category of Sets and partial functions. This should be thought of as something like the monoid of partial recursive functions, or recursively enumerable sets and partial recursive functions (both of which will yield the standard Effective topos based on Kleene realizability), or more generally we can think of it as the category associated with a typed partial combinatory algebra, as in [14], and the subsequent work in e.g. [13]. We then glue the two categories together in a variant of the comma construction: the $\mathcal{F}$-construction. Finally we construct an end category by exact completion. We explore the conditions first of all under which the construction can be carried out, and then under which the result is (locally) cartesian closed and a topos. Finally we explore the meaning of these conditions for possible forms of realizability. It turns out that the product and higher-order structure correspond to weak combinatory structures in the original category. The existence of a subobject classifier corresponds to a universal object in the category and hence to an essential untypedness in the realizability.

These results compare to the results of [13] and [2], particularly in the link between the subobject classifier and the existence of a universal object. They were inspired by the Workshop on Realizability Semantics and Applications, at FLoC in Trento at the end of June 1999. In particular we were looking for a general formulation of typed realizability in order to make precise an intuition that modified realizability could be cast in such a setting. Unfortunately, the results given here have the consequence that if such a setting exists, then this is not it!

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1 The $\mathcal{F}$-construction

There is a simple categorical generalisation of the construction of the category of partitioned assemblies, given by a variant of the standard comma

5 construction.

Suppose $U\colon \mathbf{C} \longrightarrow \mathbf{Ptl}$ is a functor. Let $\mathcal{F}(\mathbf{C}, U)$ be the category whose objects are triples $(C, S, \sigma\colon S \longrightarrow U(C))$, and a map $f\colon (C, S, \sigma) \longrightarrow (C', S', \sigma')$ is a function $f\colon S \longrightarrow S'$ such that there exists a map $\phi\colon C \longrightarrow C'$ in $\mathbf{C}$ and

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\sigma"'] & S' \arrow[d, "\sigma'"] \\ U(C) \arrow[r, "U(\phi)"] & U(C') \end{tikzcd}$$

commutes.

1.1 NOTATION We shall always write $\mathcal{F}(\mathbf{C})$ leaving U understood. Instead of $(C, S, \sigma\colon S \longrightarrow U(C))$, we shall write a typical object of $\mathcal{F}(\mathbf{C})$ as $\sigma\colon S \longrightarrow U(C)$, using the fact that we can recover C from the notation $U(C)$. Finally, we shall write morphisms as pairs (f, ϕ). This is redundant in that the equality between morphisms is based only on the first component, but we shall need to use the second in some of our constructions.

1.2 EXAMPLE As part of the construction of a standard realizability topos, $\mathbf{C}$ can be taken to be the monoid of representable partial endofunctions on the partial combinatory algebra in use. In this case $\mathcal{F}(\mathbf{C})$ is the category of partitioned assemblies (the projective objects) in the associated realizability topos. In particular, if we take $\mathbf{C}$ to be the monoid of partial recursive functions on N, then we will get the separated objects of the classical effective topos.

This construction can be seen as realizing an idea of functions based on an intensional notion of computation (given by the existence of ϕ). The category $\mathcal{F}(\mathbf{C})$ always has equalizers, and we shall see that weak conditions on $\mathbf{C}$ ensure products in $\mathcal{F}(\mathbf{C})$. Similarly, weak conditions on $\mathbf{C}$ ensure that the exact completion $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ is locally cartesian closed.

Like a comma category, $\mathcal{F}(\mathbf{C})$ comes equipped with a number of functors.

Let $\mathbf{C}_{\mathrm{t}}$ be the inverse image along U of the subcategory of total functions, then there is a full functor $\mathsf{Y}\colon \mathbf{C}_{\mathrm{t}} \longrightarrow \mathcal{F}(\mathbf{C})$ defined by

$$[C \xrightarrow{f} C'] \longmapsto [(C, U(C), \mathrm{id}) \xrightarrow{(U(f), f)} (C', U(C'), \mathrm{id})]$$

or

$$[C \xrightarrow{f} C'] \longmapsto [(C \xrightarrow{\mathrm{id}} U(C)) \xrightarrow{(U(f),f)} (C' \xrightarrow{\mathrm{id}} U(C'))]$$

This becomes full and faithful when U is faithful.

Because of the existence condition in the definition of morphisms, there is no forgetful functor $\mathcal{F}(\mathbf{C}) \to \mathbf{C}$, however there is one $\mathcal{F}(\mathbf{C}) \to \mathbf{Set}$. More significantly, let C be an arbitrary object of $\mathbf{C}$, and $x: 1 \longrightarrow U(C)$ an arbitrary element of $U(C)$, then there is a full embedding $\nabla_{C,x}: \mathbf{Set} \longrightarrow \mathcal{F}(\mathbf{C})$ defined by

$$\nabla: [S \xrightarrow{f} S'] \longmapsto [(S \xrightarrow{x \circ !} U(C)) \xrightarrow{(f,U(\mathrm{id}))} (S' \xrightarrow{x \circ !} U(C))]$$

This definition is quite robust. $\nabla_{C,x}$ is equivalent to $\nabla_{C',x'}$ provided there is a morphism $\phi: C \longrightarrow C'$ such that $\phi x = x'$ and conversely.

Because of the existence condition in the definition of maps of $\mathcal{F}(\mathbf{C})$, it is clear that $\mathcal{F}(\mathbf{C})$ is equivalent to $\mathcal{F}(U[\mathbf{C}])$ where $U[\mathbf{C}]$ is the quotient of $\mathbf{C}$ with two maps identified when they have the same value under U (in other words, the category sitting in the middle of the (full and identity on objects)/faithful factorisation of U).

1.3 NOTATION In order to make things less cluttered, from now on we shall write $\overline{(\ )}$ for the functor U, so $U(C) = \overline{C}$ and $U(\delta) = \overline{\delta}$.

2 Exact Completions

Our construction procedes in two stages. We begin by constructing a base category using the $\mathcal{F}$-construction, and then we construct a better-behaved category from that using an exact completion. In other words our final category is a free exact category on a category obtained by means of the $\mathcal{F}$-construction.

Our results, then, rely on the fundamental property of an exact completion (cf. [6]): Given an exact category $\mathbf{A}$, let $\mathbf{P}$ be the full subcategory on the regular projectives of $\mathbf{A}$. Then $\mathbf{A}$ is an exact completion of a category with finite limits if and only if $\mathbf{P}$ is closed under finite limits and each object in $\mathbf{A}$ is covered by a regular projective (i.e. for every A in $\mathbf{A}$ there is a regular epi $P \longrightarrow\!\!\!\!\triangleright\, A$ from a regular projective). When this is the case, $\mathbf{A}$ is the exact completion of $\mathbf{P}$.

The crucial point here is that the base category of projectives, $\mathbf{P}$, which in our case is going to be $\mathcal{F}(\mathbf{C})$ must be left exact, and in the next section we explore conditions under which this is so.

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3 Finite limits

First, we observe that $\mathcal{F}(\mathbf{C})$ always has equalisers. This reduces the question to when $\mathcal{F}(\mathbf{C})$ has products. It is fairly easy to see when $\mathcal{F}(\mathbf{C})$ has a terminal object: either $\overline{C}$ is empty for all C, or else for some object I of $\mathbf{C}$, $\overline{I}$ has a distinguished element ψ, such that for each object C of $\mathbf{C}$ there is a map $t_C\colon C \longrightarrow I$ such that $\overline{t_C}$ is the constant map on ψ. However, characterising products seems more difficult.

Fortunately, in the cases we are interested in we can ask a slightly easier question. Suppose $\mathbf{C}$ is a monoidal category, and $\overline{(\ )}$ is a monoidal functor. Then $\mathcal{F}(\mathbf{C})$ has a candidate for a monoidal structure. The unit is given by $\psi\colon 1 \longrightarrow \overline{I}$, and the tensor by $(f\colon X \longrightarrow \overline{C}) \otimes (g\colon Y \longrightarrow \overline{D}) = \theta(f \times g)\colon X \times Y \longrightarrow \overline{C \otimes D}$, where ψ and $\theta\colon \overline{C} \times \overline{D} \longrightarrow \overline{C \otimes D}$ are the maps given by the monoidal structure of $\overline{(\ )}$. These definitions give valid objects of $\mathcal{F}(\mathbf{C})$ if and only if ψ and θ are total. In this case, the resulting structure is indeed monoidal. The verification is almost, but not quite, straightforward. The definition of functoriality of the tensor product and most of the equations resolve into squares which split into top and bottom halves. The bottom half commutes by naturality and coherence of the monoidal structure on $\overline{(\ )}$. The arguments to show the top halves commute involve use of totality of various operations. These observations extend without problems to the symmetric case.

The simpler question we ask is: when is this monoidal structure actually a product? This simplification is not without cost. We noted above that $\mathcal{F}(\mathbf{C})$ is equivalent to $\mathcal{F}(\overline{\mathbf{C}})$ where $\overline{\mathbf{C}}$ is the quotient of $\mathbf{C}$ with two maps identified when they have the same value under $\overline{(\ )}$. This suggests that without loss of generality we can take $\overline{(\ )}$ to be faithful. This is not, unfortunately, the case. The problem is that the monoidal structure on $\mathbf{C}$ does not necessarily transfer to one on $\overline{\mathbf{C}}$. Unless θ is iso, the monoidal tensor does not necessarily respect equivalence of maps. But, unless $\overline{(\ )}$ is faithful we cannot completely reflect properties of $\mathcal{F}(\mathbf{C})$ back into properties of $\mathbf{C}$. This explains why in general we prove properties up to the functor $\overline{(\ )}$, leaving the cleaner and perhaps more interesting case where $\overline{(\ )}$ is faithful to corollaries. This is first evident in the characterisation of when the monoidal unit on $\mathcal{F}(\mathbf{C})$ is terminal.

3.1 LEMMA $\quad$ *$\psi\colon 1 \longrightarrow \overline{I}$ is terminal in $\mathcal{F}(\mathbf{C})$ if and only if for each object*

C of $\mathbf{C}$ there is a map $t_C : C \longrightarrow I$ such that $\overline{t_C} = \psi \circ !$.

$$\begin{tikzcd}
\overline{C} \arrow[r, "!"] \arrow[d, "\mathrm{id}"'] & 1 \arrow[d, "\psi"] \\
\overline{C} \arrow[r, "\overline{t_C}"] & \overline{I}
\end{tikzcd}$$

Proof If $\psi : 1 \longrightarrow \overline{I}$ is terminal, then we obtain t_C by considering the terminal map from $\mathrm{id} : \overline{C} \longrightarrow \overline{C}$ to ψ (the diagram is as above). Conversely, given such a family of maps, ψ is weakly terminal because for any $f : X \longrightarrow \overline{C}$, we have the following diagram (note that the upper triangle commutes because f is total).

$$\begin{tikzcd}
X \arrow[r, "!"] \arrow[d, "f"'] & 1 \arrow[d, "\psi"] \\
\overline{C} \arrow[r, "\overline{t_C}"] \arrow[ur, "!"] & \overline{I}
\end{tikzcd}$$

However, maps into ψ are unique, when they exist, because maps into 1 are. This establishes that ψ is terminal. $\square$

Now, if the unit of a monoidal category is terminal, then there are candidates for left and right projections from the tensor:

$$\pi_{0,XY} = \rho_X(\mathrm{id}_X \otimes t_Y) : X \otimes Y \longrightarrow X \qquad \pi_{1,XY} = \lambda_Y(t_X \otimes \mathrm{id}_Y) : X \otimes Y \longrightarrow Y$$

Reading this out in $\mathcal{F}(\mathbf{C})$ the left projection is

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_0"] \arrow[d, "f \times g"'] & X \arrow[d, "f"] \\
\overline{C} \times \overline{D} \arrow[r, "\pi_0"] \arrow[d, "\theta"'] & \overline{C} \arrow[d, "\mathrm{id}"] \\
\overline{C \otimes D} \arrow[r, "\overline{\rho_C(\mathrm{id}_C \otimes t_D)}"] & \overline{C}
\end{tikzcd}$$

and the right projection is similar. Given the monoidal structure, there is no real need to check that this diagram commutes. However, here is an elementary proof:

$$\begin{tikzcd}
X \times Y \arrow[rr, "\pi_0"] \arrow[d, "f \times g"'] & & X \arrow[d, "f"] \\
\overline{C} \times \overline{D} \arrow[r, "\mathrm{id} \times !"] \arrow[dd, "\theta"'] & \overline{C} \times 1 \arrow[r, "\rho"] \arrow[d, "\mathrm{id} \times \psi"] & \overline{C} \arrow[dd, "\mathrm{id}"] \\
 & \overline{C} \times \overline{I} \arrow[d, "\theta"'] & \\
\overline{C \otimes D} \arrow[r, "\overline{\mathrm{id} \otimes t_D}"] & \overline{C \otimes I} \arrow[r, "\rho"] & \overline{C}
\end{tikzcd}$$

In this diagram the lower left hand rectangle commutes because $\psi ! = \overline{t}$, and then by naturality of θ. The lower right rectangle commutes by coherence of the monoidal functor $\overline{(\ )}$.

We can now ask the question of when the monoidal tensor is a product, in the sense that these projections together form a product cone.

3.2 LEMMA *In the case that $\psi: 1 \longrightarrow \overline{I}$ is terminal in $\mathcal{F}(\mathbf{C})$, then the candidates for projections above form product cones if and only if for each object c of $\mathbf{C}$ there is a map $d_C: C \longrightarrow C \otimes C$ such that $\overline{d_C} = \theta \circ \Delta_{\overline{C}}$, where $\Delta_{\overline{C}}: \overline{C} \longrightarrow \overline{C} \times \overline{C}$ is the ordinary cartesian diagonal.*

Proof First, suppose that the tensor is cartesian product. Then the tensor of $\mathrm{id}: \overline{C} \longrightarrow \overline{C}$ with itself is $\theta: \overline{C} \times \overline{C} \longrightarrow \overline{C \otimes C}$. This must have a diagonal

$$\begin{tikzcd}
\overline{C} \arrow[r, "D"] \arrow[d, "\mathrm{id}"'] & \overline{C} \times \overline{C} \arrow[d, "\theta"] \\
\overline{C} \arrow[r, "\overline{d_C}"] & \overline{C \otimes C}
\end{tikzcd}$$

Composing with the projections we see that D must be the diagonal $\Delta_{\overline{C}}$, and the square then yields $\overline{d_C} = \theta \circ \Delta_{\overline{C}}$, as required.
For the converse, suppose we have two maps

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d] & X \arrow[d, "f"] \\
\overline{C} \arrow[r, "\overline{a}"] & \overline{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "k"] \arrow[d] & Y \arrow[d, "g"] \\
\overline{C} \arrow[r, "\overline{b}"] & \overline{B}
\end{tikzcd}$$

then we can form the pairing

$$\begin{tikzcd}
Z \arrow[r, "\Delta"] \arrow[d] & Z \times Z \arrow[r, "h \times k"] \arrow[d] & X \times Y \arrow[d, "f \times g"] \\
\overline{C} \arrow[r, "\Delta"] \arrow[d] & \overline{C} \times \overline{C} \arrow[r, "\overline{a} \times \overline{b}"] \arrow[d, "\theta"'] & \overline{A} \times \overline{B} \arrow[d, "\theta"] \\
\overline{C} \arrow[r, "\overline{d_C}"] & \overline{C \otimes C} \arrow[r, "\overline{a \otimes b}"] & \overline{A \otimes B}
\end{tikzcd}$$

given by composing the obvious "diagonal" on $Z \longrightarrow \overline{C}$ (the left-hand half of the diagram) with the tensor product. Each small square in this diagram commutes, with that on the lower left commuting by the given property of d_C. We can now compose with the projections. We are only interested in the map along the top edge, and this is pairing followed by projection in the cartesian category **Set**, hence is h or k. Thus pairing followed by projection behaves as expected, and the tensor is at least a weak product. As with the terminal object, the weak structure is strong because maps in $F(\mathbf{C})$ are identified if and only if their first components are equal, and because the topmost edge of our diagram is a real product. $\square$

Note that Δ and θ are natural considered as transformations between functors $\mathbf{C} \longrightarrow \mathbf{Ptl}$, hence $\overline{d_C}$ is natural in C. Hence, if $\overline{(\ )}$ is faithful, then d_C itself is natural in C. However, although ψ is natural in C, ! is only natural in the subcategory of total maps. Whence $\overline{t_C}$ (and hence t_C, if $\overline{(\ )}$ is faithful) is natural only in the category of total maps in $\mathbf{C}$.

11 Moreover, it is not necessarily the case that $\overline{I}$ is isomorphic to 1, or that $\overline{X \otimes Y}$ is isomorphic to $\overline{X} \times \overline{Y}$. However:

3.3 LEMMA *There is a map $e: X \otimes Y \longrightarrow X \otimes Y$ such that $\overline{e}$ is idempotent split by*

$$\overline{X \otimes Y} \xrightarrow{(\overline{\pi_0}, \overline{\pi_1})} \overline{X} \times \overline{Y} \xrightarrow{\theta} \overline{X \otimes Y}$$

where $\pi_0 = \rho(\mathrm{id} \otimes t)$ and $\pi_1 = \lambda(t \otimes \mathrm{id})$.

Proof It is easy to check that $(\overline{\pi_0}, \overline{\pi_1}) \circ \theta: \overline{X} \times \overline{Y} \longrightarrow \overline{X} \times \overline{Y}$ is the identity. It is therefore enough to check that $\theta \circ (\overline{\pi_0}, \overline{\pi_1})$ is in the image of $\overline{(\ )}$. But $\theta \circ (\overline{\pi_0}, \overline{\pi_1}) = \theta \circ (\overline{\pi_0} \times \overline{\pi_1}) \circ \Delta = (\overline{\pi_0 \otimes \pi_1}) \circ \theta \circ \Delta = (\overline{\pi_0 \otimes \pi_1}) \circ \overline{d}$ $\square$

Similarly t_I is an endomorphism on I, such that $\overline{t_I}$ is split by

$$\overline{I} \xrightarrow{\ \ !\ \ } 1 \xrightarrow{\ \ \psi\ \ } \overline{I}$$

Recapping, we have amongst other things, established a criterion for $\mathcal{F}(\mathbf{C})$ to be left exact.

3.4 LEMMA *If $\mathbf{C}$ is a symmetric monoidal category and $\overline{(\ )}$ a faithful symmetric monoidal functor $\mathbf{C} \longrightarrow \mathbf{Ptl}$, for which the structural maps $\psi: 1 \longrightarrow \overline{I}$ and $\theta_{C,D}: \overline{C} \times \overline{D} \longrightarrow \overline{C \otimes D}$ are total, then $\mathcal{F}(\mathbf{C})$ carries a symmetric monoidal structure. This is a product structure, i.e. the unit is terminal, and the monoidal product together with projections defined from terminal maps and monoidal structure forms product cones, if and only if for each object C of $\mathbf{C}$ there are maps $t_C: C \longrightarrow I$ and $d_C: C \longrightarrow C \otimes C$ such that $\overline{t_C} = \psi!$ and $\overline{d_C} = \theta_{C,C}\Delta_{\overline{C}}$. In addition, $\overline{d_C}$ is natural in C (though $\overline{t_C}$ is not).*

If $\mathcal{F}(\mathbf{C})$ is left exact, then we can take its exact completion. This is our candidate for a topos. In the next two sections we investigate the conditions under which this category is cartesian closed and has a subobject classifier. It turns out to be simpler to deal with the subobject classifier first.

4 Subobject classifiers and universal objects

In this section we investigate the connection between the existence of a subobject classifier in $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ and universal objects in $\mathbf{C}$. The connection is tight when U is faithful, but can be deduced immediately from a more technical statement which holds in general.

4.1 DEFINITION The category $\mathbf{C}$ has a *universal object* W if each object C of $\mathbf{C}$ is a retract of W.

4.2 PROPOSITION *If the category $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ has a subobject classifier, then there is an object W of $\mathbf{C}$, such that for each object C of $\mathbf{C}$ there are morphisms $\gamma\colon C \longrightarrow W$ and $\delta\colon W \longrightarrow C$ such that $\overline{\delta}\overline{\gamma}$ is the identity on $\overline{C}$.*

Intuitively, modulo U, W is universal in $\mathbf{C}$. If U is faithful, then this immediately implies that $\delta\gamma = \mathrm{id}_C$.

4.3 COROLLARY *If the category $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ has a subobject classifier, and the functor U is faithful, then the category $\mathbf{C}$ has a universal object.*

This result is closely connected to one in [13, 1] obtained for the subcategory of an exact completion as above which is the regular completion of $\mathbf{P}$, see [4]. The contexts seem rather different, though indeed they are not.

Our proof builds on previous analysis of subobject classifiers in exact completions. The following is a slight variant of Menni [15].

4.4 DEFINITION A map $u\colon W \longrightarrow V$ is a *(weakly) weak proof classifier* if every map in the category appears as weakly equivalent to a (weak) pullback of u: i.e. for every map $a\colon X \longrightarrow A$ there is a diagram

$$\begin{tikzcd}
X \arrow[r, shift left, "h"] \arrow[dr, "a"'] & X' \arrow[l, shift left, "k"] \arrow[r] \arrow[d, "a'"] & W \arrow[d, "u"] \\
& A \arrow[r, "f"] & V
\end{tikzcd} \tag{1}$$

where the square is a (weak) pullback, and the triangles commute.

13 In an exact category $\mathbf{A}$ where every object is covered by a regular projective, a weakly weak proof classifier is what can be traced directly in the full subcategory $\mathbf{P}$ of projectives when $\mathbf{A}$ has a subobject classifier. If in addition $\mathbf{P}$ is a left exact subcategory (as when $\mathbf{A}$ is its exact completion), then a weakly weak proof classifier is actually a weak proof classifier in the sense of Menni (the weak pullback in the definition can be taken to be a pullback).

We do not need an explicit construction of a weak proof classifier, it is not, however, hard to give one:

4.5 LEMMA *Suppose that $\mathbf{P}_{\mathrm{ex}}$, the exact completion of a left exact category $\mathbf{P}$, has a subobject classifier. Let $\pi: V \longrightarrow \Omega$ be a cover of Ω by a projective, $\pi^*\mathrm{true}: Y \longrightarrow V$, the pullback of the classifier $\mathrm{true}: 1 \longrightarrow \Omega$ along π and $\phi: W \longrightarrow Y$ a cover of Y by a projective. Then $(\pi^*\mathrm{true})\phi: W \longrightarrow V$ is a weak proof classifier.*

We will also need a further technical result, establishing a factorisation property which generalises a standard lemma for subobject classifiers, and is best seen in the abstract:

4.6 LEMMA *Suppose that $u: W \longrightarrow V$ is a (weakly) weak proof classifier, and that $a: X \longrightarrow A$ is an arbitrary map. The (weakly) weak proof classifier produces a diagram*

$$\begin{tikzcd}
X \arrow[r, shift left, "h"] \arrow[dr, "a"'] & X' \arrow[l, shift left, "k"] \arrow[r] \arrow[d, "a'"] & W \arrow[d, "u"] \\
& A \arrow[r, "f"] & V
\end{tikzcd}$$

Suppose, now that $b: Y \longrightarrow A$ makes f true, in the sense that $f \circ b$ factors through u, then b factors through a.

Proof The square

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "b"'] & W \arrow[d, "u"] \\
A \arrow[r, "f"] & V
\end{tikzcd}$$

yields a map $l: Y \longrightarrow X'$ such that $b = a' \circ l$. Therefore $b = a \circ (k \circ l)$. $\square$

Proof of 4.2 We begin by following the recipe given in lemma 4.5. Suppose that $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ has a subobject classifier, and consider a projective cover $(v\colon P \longrightarrow \overline{V}) \longrightarrow\!\!\triangleright\ \Omega$ of Ω. Cover the pullback of true: $1 \longrightarrow \Omega$ along this by a projective, say $(w\colon Q \longrightarrow \overline{W})$, obtaining a map

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "w"'] & P \arrow[d, "v"] \\ \overline{W} \arrow[r, harpoon, "\phi"] & \overline{V} \end{tikzcd} \tag{2}$$

in $\mathcal{F}(\mathbf{C})$, which is a weak proof classifier.

Now take an arbitrary object C of $\mathbf{C}$. Consider the map

$$\iota\colon \mathsf{Y}(C) = (\overline{C} \xrightarrow{\mathrm{id}} \overline{C}) \longrightarrow \nabla(\overline{C}) = (\overline{C} \xrightarrow{\overline{t_C}} \overline{I})$$

defined as

$$\begin{tikzcd} \overline{C} \arrow[r, "\mathrm{id}_{\overline{C}}"] \arrow[d, "\overline{\mathrm{id}_C}"'] & \overline{C} \arrow[d, "\overline{t_C}"] \\ \overline{C} \arrow[r, "\overline{t_C}"] & \overline{I} \end{tikzcd} \tag{3}$$

Since (f, ϕ) is a weak proof classifier, this map appears in a commutative square:

$$\begin{tikzcd} \mathsf{Y}(C) \arrow[r] \arrow[d, "\iota"'] & (w\colon Q \to \overline{W}) \arrow[d, "(f,\phi)"] \\ \nabla(\overline{C}) \arrow[r, "(k,\kappa)"] & (v\colon P \to \overline{V}) \end{tikzcd} \tag{4}$$

15 Reading this out in **Ptl**, we have a commutative diagram of sets

$$\begin{tikzcd}
\overline{C} \arrow[rr, "g"] \arrow[dr, "\mathrm{id}"] \arrow[dd, "\mathrm{id}"'] & & Q \arrow[dr, "f"] \arrow[dd, "w"' near start] & \\
& \overline{C} \arrow[rr, "k" near start] \arrow[dd, "\overline{t}"' near start] & & P \arrow[dd, "v"] \\
\overline{C} \arrow[rr, harpoon, "\overline{\gamma}" near end] \arrow[dr, "\overline{t}"'] & & \overline{W} \arrow[dr, "\phi"] & \\
& \overline{I} \arrow[rr, harpoon, "\kappa"] & & \overline{V}
\end{tikzcd}$$

In this diagram, g and w are both total, hence so is $\overline{\gamma}$, which is equal to their composite. So $\overline{\gamma}\colon \overline{C} \longrightarrow \overline{W}$ is an element of $\mathcal{F}(\mathbf{C})$. Moreover we have a morphism $(\mathrm{id}, \overline{t_W})\colon (\overline{\gamma}\colon \overline{C} \to \overline{W}) \longrightarrow (\overline{t_C}\colon \overline{C} \to \overline{I})$. We now check that the following square commutes:

$$\begin{tikzcd}
\overline{\gamma} \arrow[r, "(\mathrm{id}, \overline{t_W})"] \arrow[d, "(g, \mathrm{id})"'] & \nabla(\overline{C}) \arrow[d, "(k, \kappa)"] \\
w \arrow[r, "(f, \kappa)"] & v
\end{tikzcd}$$

In more detail, this is the cube:

$$\begin{tikzcd}
\overline{C} \arrow[rr, "\mathrm{id}"] \arrow[dr, "g"] \arrow[dd, "\overline{\gamma}"'] & & \overline{C} \arrow[dr, "k"] \arrow[dd, "\overline{t}"' near start] & \\
& Q \arrow[rr, "f" near start] \arrow[dd, "w"' near start] & & P \arrow[dd, "v"] \\
\overline{W} \arrow[rr, "\overline{t_W}" near end] \arrow[dr, "\mathrm{id}"'] & & \overline{I} \arrow[dr, "\kappa"] & \\
& \overline{W} \arrow[rr, harpoon, "\kappa"] & & \overline{V}
\end{tikzcd}$$

Here, the four side squares and the top square commute. The bottom square might not commute, and does not need to.

Hence by lemma 4.6, $(\mathrm{id}_{\overline{C}}, t_{\overline{W}})$ factors through $(\mathrm{id}_{\overline{C}}, t_{\overline{C}})$, producing a prism:

$$
\begin{tikzcd}
\overline{C} \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] \arrow[dd, "\overline{\gamma}"'] & & \overline{C} \arrow[dd, "\overline{t}"] \\
& \overline{C} \arrow[ur, "\mathrm{id}"'] \arrow[dd, "\mathrm{id}"] & \\
\overline{W} \arrow[rr, "\overline{t}"] \arrow[dr, "\overline{\delta}"'] & & \overline{I} \\
& \overline{C} \arrow[ur, "\overline{t}"'] &
\end{tikzcd}
$$

and thus a map δ in $\mathbf{C}$, such that $\overline{\delta}\overline{\gamma} = \mathrm{id}$. $\square$

5 Function spaces

In this section we deal with conditions for the cartesian closure of $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$. We make the running assumption that $\mathbf{C}$ is a symmetric monoidal category, and $U\colon \mathbf{C} \longrightarrow \mathbf{Ptl}$ a symmetric monoidal functor satisfying the conditions of Lemma 3.4, so that $\mathcal{F}(\mathbf{C})$ is a left exact category with product structure constructed from the monoidal structure on $\mathbf{C}$. We shall abuse the structure and refer to a map f in $\mathbf{C}$ as *total* just when its image under U, $\overline{f}$, is total.

As in section 4, our work builds heavily on previous work on properties of exact completions. One of the major lessons of [7] is that in this context it is easier to deal with local cartesian closure, than simple cartesian closure. So it is an important fact that exact completion is a local construction.

5.1 LEMMA *Let $\mathbf{P}$ be a left exact category with exact completion $\mathbf{A}$. Then for any object P of $\mathbf{P}$, the slice $\mathbf{A}/P$ is the exact completion of $\mathbf{P}/P$.*

We shall use this in combination with the following facts about cartesian closure of exact completions.

5.2 LEMMA *Let $\mathbf{P}$ be a left exact category with exact completion $\mathbf{A}$. If $\mathbf{A}$ is cartesian closed, then for any objects P and Q of $\mathbf{P}$, there is a weak evaluation $\epsilon\colon F \times P \longrightarrow Q$ from P to Q in $\mathbf{P}$, i.e. any map $\phi\colon X \times P \longrightarrow Q$ can be expressed as $\epsilon \circ (f \times \mathrm{id}_P)$ for some $f\colon X \longrightarrow F$ (here all of the last part of the statement takes place in $\mathbf{P}$).*

17 **Proof** Take F to be a projective cover of the exponential Q^P in $\mathbf{A}$. $\square$

We shall need to deal with a monoidal structure that approximates a product. First some notation.

5.3 NOTATION The structure on $\mathbf{C}$ gives operations which we can loosely think of as pairing and projections, and which we shall write:

$$\langle f, g\rangle = (f \otimes g) \circ d_Z \colon Z \longrightarrow X \otimes Y$$

$$\pi_0 = \rho_X \circ (\mathrm{id}_X \otimes t_Y) \colon X \otimes Y \longrightarrow X \qquad \pi_1 = \lambda_Y \circ (t_X \otimes \mathrm{id}_Y) \colon X \otimes Y \longrightarrow Y$$

We shall use (a, b), p_0 and p_1 for the usual pairing and projections from a categorical product.

Later, we shall need some technical properties of these operations.

5.4 LEMMA *The following properties hold of pairing and projection in* $\mathbf{C}$*:*

(i) $\overline{\langle a, b\rangle} = \theta \circ (\overline{a}, \overline{b})$

(ii) $\overline{\pi_i} \circ \theta = p_i$

(iii) $\overline{\pi_0 \circ \langle a, b\rangle} = \overline{a}$ *and* $\overline{\pi_1 \circ \langle a, b\rangle} = \overline{b}$.

(iv) $\pi_0 \colon X \otimes Y \longrightarrow X$ *is natural in* X, *and* $\pi_1 \colon X \otimes Y \longrightarrow Y$ *in* Y.

(v) $\pi_1 \otimes \mathrm{id}_C = \pi_1 \circ \alpha \colon (A \otimes B) \otimes C \longrightarrow B \otimes C$, *and similarly for other projections, where* $\alpha \colon (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ *is the associativity isomorphism of the monoidal category.*

Proof Parts (i) and (ii) are easy diagram chases using the defining properties of d_C and t_C respectively. Part (iii) follows since, e.g. $\overline{\pi_0 \circ \langle a, b\rangle} = \overline{\pi_0} \circ \theta \circ (\overline{a}, \overline{b}) = p_0 \circ (\overline{a}, \overline{b}) = \overline{a}$. Finally, parts (iv) and (v) follow simply from properties of the monoidal structure. $\square$

Returning to our main interest, if

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "\sigma"'] & Y \arrow[d, "\sigma'"] \\
\overline{C} \arrow[r, "\overline{g}"] & \overline{D}
\end{tikzcd}$$

a morphism in $\mathcal{F}(\mathbf{C})$, then we can replace g by any morphism f such that $\overline{f}$ extends $\overline{g}$. It follows that we will not be able to prove that something we have constructed in $\mathbf{C}$ gives an arbitrary partial function g, only that it extends g. Moreover, we have seen that our monoidal structure is not a product, but the product is related by retraction. This motivates the following definition: *18*

5.5 DEFINITION Suppose $\mathbf{C}$ is a monoidal category equipped with a functor $\overline{(\ )}$ into $\mathbf{Ptl}$, together with families of maps $t_C: C \longrightarrow I$ and $d_C: C \longrightarrow C \otimes C$, as in our standard structure. Then we say that a morphism $f: A \longrightarrow B$ *extends* a morphism $g: A \longrightarrow B$ ($g \subseteq f: A \longrightarrow B$) if $\overline{g} \subseteq \overline{f}$. We now say that a map $\epsilon: F \otimes C \longrightarrow C'$ is a *weak partial evaluation* from C to C' if for every map $\phi: X \otimes C \longrightarrow C'$ there is a total $f: X \longrightarrow F$, such that $\epsilon \circ (f \otimes \mathrm{id}_C) \circ e$ extends $\phi \circ e$, where $e: X \otimes C \longrightarrow X \otimes C$ is the "η-retraction" for pairing $e = \langle \pi_0, \pi_1 \rangle$.

5.6 PROPOSITION *Suppose* $\mathbf{C}$ *and* $U: \mathbf{C} \longrightarrow \mathbf{Ptl}$ *satisfy our running assumptions, then if the exact completion of* $\mathcal{F}(\mathbf{C})$ *is locally cartesian closed, then for any pair of objects* C *and* C' *of* $\mathbf{C}$*, there is an object* F *of* $\mathbf{C}$ *and a map* $\epsilon: F \otimes C \longrightarrow C'$*, such that for any map* $\phi: X \otimes C \longrightarrow C'$*, there is a map* $f: X \longrightarrow F$ *such that* $\overline{f}$ *is total and* $\overline{\epsilon \circ (f \otimes \mathrm{id}_C) \circ e}$ *extends* $\overline{\phi \circ e}: X \otimes C \longrightarrow C'$*, where* $e: X \otimes C \longrightarrow X \otimes C$ *is the* η*-retraction for pairing* $e = \langle \pi_0, \pi_1 \rangle$*, as before.*

5.7 COROLLARY *If in the above the functor* U *is faithful, then the exact completion of* $\mathcal{F}(\mathbf{C})$ *is locally cartesian closed if and only if* $\mathbf{C}$ *has weak partial evaluations.*

Proof of 5.6 Given objects C and C' in $\mathbf{C}$, consider the set $\mathcal{D}_C$ of subfunctions of the identity on $\overline{C}$. This gives rise to an object $\nabla(\mathcal{D}_C)$ of $\mathcal{F}(\mathbf{C})$, as in section 1:

$$\psi \circ !: \mathcal{D}_C \longrightarrow 1 \longrightarrow \overline{I}$$

Given $d \in \mathcal{D}_C$, let D_d be the set of fixpoints of d, the domain of d. Further, let

$$j: \textstyle\sum_{d \in \mathcal{D}_C} D_d \longrightarrow \overline{C}.$$

be the sum of the inclusions of the sets of fixpoints in $\overline{C}$. We shall treat this as an element of $\mathcal{F}(\mathbf{C})$, and call it J.

19 We shall work in the slices of $\mathcal{F}(\mathbf{C})$ and $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ over $\nabla(\mathcal{D}_C)$. In particular an object of $\mathcal{F}(\mathbf{C})/\nabla(\mathcal{D}_C)$ is given by

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dd] & \mathcal{D}_C \arrow[d, "!"'] \\
 & 1 \arrow[d, "\psi"'] \\
\overline{D} \arrow[r, harpoon, "\overline{\alpha}"] & \overline{I}
\end{tikzcd}$$

Note that $\overline{\alpha}$ is a subfunction of $\overline{t_D}$, and hence that without loss of generality α can be taken to be t_D. Note further that commutativity of the diagram then imposes no real constraint on f. Thus an object of $\mathcal{F}(\mathbf{C})/\nabla(\mathcal{D}_C)$ is given by an object $X \longrightarrow \overline{D}$ of $\mathcal{F}(\mathbf{C})$, together with a function $f\colon X \longrightarrow \mathcal{D}_C$.
In particular, if $i\colon \sum_{d\in\mathcal{D}_C} D_d \longrightarrow \mathcal{D}_C$ is the map giving the indices of the elements of the sum, we have an object Q of $\mathcal{F}(\mathbf{C})/\nabla(\mathcal{D}_C)$

$$\begin{tikzcd}
\sum_{d\in\mathcal{D}_C} D_d \arrow[r, "i"] \arrow[d, "j"'] & \mathcal{D}_C \arrow[d, "\psi\circ!"] \\
\overline{C} \arrow[r, "\overline{t}"] & \overline{I}
\end{tikzcd}$$

We also have a constant object P given by

$$\begin{tikzcd}
\overline{C'} \times \mathcal{D}_C \arrow[r, "\pi_1"] \arrow[d, "\theta\circ(\overline{C}\times(\psi\circ!))"'] & \mathcal{D}_C \arrow[d, "\psi\circ!"] \\
\overline{C'\otimes I} \arrow[r, "\overline{t}"] & \overline{I}
\end{tikzcd}$$

Since $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}/\nabla(\mathcal{D}_C)$ is cartesian closed, and, by 5.1, the exact completion of its category of projectives $\mathcal{F}(\mathbf{C})/\nabla(\mathcal{D}_C)$, we can apply lemma 5.2 to obtain a weak evaluation $F \times Q \longrightarrow P$. F is given by an object $\omega\colon E \longrightarrow \overline{F}$ of $\mathcal{F}(\mathbf{C})$ together with a map to $\nabla(\mathcal{D}_C)$.

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, "\omega"'] & \mathcal{D}_C \arrow[d, "\psi\circ!"] \\
\overline{F} \arrow[r, "\overline{t}"] & \overline{I}
\end{tikzcd}$$

The domain of the weak evaluation $(F \times Q)$ is of course the product in the fibre $\mathcal{F}(\mathbf{C})/\nabla(\mathcal{D}_C)$, which is computed by pullback in $\mathcal{F}(\mathbf{C})$. Now pullbacks in $\mathcal{F}(\mathbf{C})$ are computed by pullback in the top square and tensor in the bottom (this can easily be verified, either directly or perhaps more simply by using the construction of pullbacks from products and equalisers). This means that the underlying object of $F \times Q$ is

$$E \times_{\mathcal{D}_C} \sum_{d \in \mathcal{D}_C} D_d \rightarrowtail E \times \sum_{d \in \mathcal{D}_C} D_d \xrightarrow{\omega \times j} \overline{F} \times \overline{C} \xrightarrow{\theta} \overline{F \otimes C}$$

and the weak evaluation map into P is

$$\begin{tikzcd}
E \times_{\mathcal{D}_C} \sum_{d \in \mathcal{D}_C} D_d \arrow[r] \arrow[d] & \overline{C'} \times \mathcal{D}_C \arrow[d, "\theta(\overline{C} \times \psi\circ !)"] \\
\overline{F \otimes C} \arrow[r, harpoon, "\overline{\varepsilon}"] & \overline{C' \otimes I}
\end{tikzcd}$$

for some appropriate $\varepsilon\colon F \otimes C \longrightarrow C' \otimes I$ in $\mathbf{C}$. We set $\epsilon = \rho \circ \varepsilon$, so that $\epsilon\colon F \otimes C \longrightarrow C'$. We claim that ϵ is a weak partial evaluation up to U, as required.

Suppose $\phi\colon X \otimes C \longrightarrow C'$ is in $\mathbf{C}$. Consider $\overline{\phi} \circ \theta$. This is a partial map $\overline{X} \times \overline{C} \rightharpoonup \overline{C'}$. Now, for each x we consider the set of $c \in \overline{C}$ such that $(U(\phi) \circ \theta)(x, c)$ is defined.

Hence we obtain a function $\partial\colon \overline{X} \longrightarrow \mathcal{D}_C$, and thus a morphism into $\nabla(\mathcal{D}_C)$ in $\mathcal{F}(\mathbf{C})$:

$$\begin{tikzcd}
\overline{X} \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"'] & \mathcal{D}_C \arrow[d, "\psi\circ !"] \\
\overline{X} \arrow[r, "\overline{t}"] & \overline{I}
\end{tikzcd}$$

We shall pull back along this morphism to work in the fibre over $\mathrm{id}\colon \overline{X} \longrightarrow \overline{X}$.

21 Pulling back P along $(\partial, \overline{t})$ we get

$$
\begin{tikzcd}
\overline{X} \times \overline{C'} \arrow[r, "\pi_0"] \arrow[d, "\theta"'] & \overline{X} \arrow[d, "\mathrm{id}"] \\
\overline{X \otimes C'} \arrow[r] & \overline{X}
\end{tikzcd}
$$

and pulling back Q along $(\partial, \overline{t})$ we get

$$
\begin{tikzcd}
\overline{X} \times_{\mathcal{D}_C} \sum_{d \in \mathcal{D}_C} D_d \arrow[r, "\pi_0"] \arrow[d] & \overline{X} \arrow[d, "\mathrm{id}"] \\
\overline{X \otimes C} \arrow[r] & \overline{X}
\end{tikzcd}
$$

We now have a map in the fibre over $\mathrm{id}\colon \overline{X} \longrightarrow \overline{X}$ from $(\partial, \overline{t})^* Q$ to $(\partial, \overline{t})^* P$:

$$
\begin{tikzcd}
\overline{X} \times_{\mathcal{D}_C} \sum_{d \in \mathcal{D}_C} D_d \arrow[rr, "{(\pi_0, \phi \circ (\mathrm{id}_X \times j))}"] \arrow[dr] \arrow[dd] & & \overline{X} \times \overline{C'} \arrow[dl] \arrow[dd] \\
& \overline{X} \arrow[dd] & \\
\overline{X \otimes C} \arrow[rr, "\overline{\langle \pi_0, \phi \rangle}"] \arrow[dr] & & \overline{X \otimes C'} \arrow[dl] \\
& \overline{X} &
\end{tikzcd}
\tag{5}
$$

Here the top map is total because of the construction of the domain.
Now, the construction of the evaluation is stable under change of base, so there is a map over $\mathrm{id}\colon \overline{X} \longrightarrow \overline{X}$

$$
\begin{tikzcd}
\overline{X} \arrow[r] \arrow[d, "\mathrm{id}"'] & \overline{X} \times_{\mathcal{D}_C} E \arrow[d, "\partial^* \omega"] \\
\overline{X} \arrow[r, "\overline{f'}"] & \overline{X \otimes F}
\end{tikzcd}
$$

such that the map in diagram (5) is

$$\begin{tikzcd}
\overline{X} \times_{\mathcal{D}_C} \sum\limits_{d\in\mathcal{D}_C} D_d \arrow[r] \arrow[d] & \overline{X} \times_{\mathcal{D}_C} E \times_{\mathcal{D}_C} \sum\limits_{d\in\mathcal{D}_C} D_d \arrow[r] \arrow[d] & \overline{X} \times \overline{C'} \arrow[d, "\theta"] \\
\overline{X \otimes C} \arrow[r, "\overline{f' \otimes \mathrm{id}_C}"] & \overline{X \otimes F \otimes C} \arrow[r, "\overline{\mathrm{id}_X \otimes \epsilon}"] & \overline{X \otimes C'}
\end{tikzcd}$$

We now compose both this, and the map of diagram (5) on the right with the projection

$$\begin{tikzcd}
\overline{X} \times \overline{C'} \arrow[r, "p_1"] \arrow[d, "\theta"'] & \overline{C'} \arrow[d, "\mathrm{id}"] \\
\overline{X \otimes C'} \arrow[r, "\overline{\pi_1}"] & \overline{C'}
\end{tikzcd}$$

and on the left with

$$\begin{tikzcd}
\overline{X} \times_{\mathcal{D}_C} \sum\limits_{d\in\mathcal{D}_C} D_d \arrow[r, "\mathrm{id}"] \arrow[d] & \overline{X} \times_{\mathcal{D}_C} \sum\limits_{d\in\mathcal{D}_C} D_d \arrow[d] \\
\overline{X \otimes C} \arrow[r, "\overline{e} = \overline{\langle \pi_0, \pi_1 \rangle}"] & \overline{X \otimes C}
\end{tikzcd}$$

This gives us two commuting squares of the form

$$\begin{tikzcd}
\overline{X} \times_{\mathcal{D}_C} \sum\limits_{d\in\mathcal{D}_C} D_d \arrow[r] \arrow[d] & \overline{C'} \arrow[d, "\mathrm{id}"] \\
\overline{X \otimes C} \arrow[r] & \overline{C'}
\end{tikzcd}$$

which both represent the same map in $\mathcal{F}(\mathbf{C})$. The two lower routes are therefore equal. Looking at the one derived from our original map, we have

$$\begin{tikzcd}
\overline{X} \times_{\mathcal{D}_C} \sum\limits_{d\in\mathcal{D}_C} D_d \arrow[rrr] \arrow[d] & & & \overline{C'} \arrow[d, "\mathrm{id}"] \\
\overline{X \otimes C} \arrow[r, "(\overline{\pi_0}, \overline{\pi_1})"] & \overline{X} \times \overline{C} \arrow[r, "\theta"] & \overline{X \otimes C} \arrow[r, "\overline{\pi_1 \circ \langle \pi_0, \phi \rangle}"] & \overline{C'}
\end{tikzcd}$$

23 and at the other:

$$\begin{tikzcd}
\overline{X} \times_{\mathcal{D}_C} \sum_{d\in\mathcal{D}_C} D_d \arrow[rrr] \arrow[d] & & & \overline{C'} \arrow[d, "\mathrm{id}"] \\
\overline{X\otimes C} \arrow[r, "(\overline{\pi_0},\overline{\pi_1})"] & \overline{X}\times\overline{C} \arrow[r, "\theta"] & \overline{X\otimes C} \arrow[r, "\overline{\pi_1\circ(\mathrm{id}_X\otimes\epsilon)\circ(f'\otimes\mathrm{id}_C)}"] & \overline{C'}
\end{tikzcd}$$

Now, by lemma 5.4, $\overline{\pi_1\circ\langle\pi_0,\phi\rangle} = \overline{\phi}$, and $\pi_1\circ(\mathrm{id}_X\otimes\epsilon)\circ(f'\otimes\mathrm{id}_C) = \epsilon\circ(\pi_1\otimes\mathrm{id}_C)\circ(f'\otimes\mathrm{id}_C)$. So if we set $f=\pi_1\circ f'$, then $\pi_1\circ(\mathrm{id}_X\otimes\epsilon)\circ(f'\otimes\mathrm{id}_C)$ simplifies to $\epsilon\circ(f\otimes\mathrm{id}_C)$.
Now, by construction, the composite from $\overline{X}\times_{\mathcal{D}_C}\sum_{d\in\mathcal{D}_C} D_d$ to $\overline{X}\times\overline{C}$ maps onto the domain of $\overline{\phi}\circ\theta$. It follows that $\overline{\epsilon\circ(f\otimes\mathrm{id}_C)}\circ\theta$ extends $\overline{\phi}\circ\theta$, and hence that $\overline{\epsilon\circ(f\otimes\mathrm{id}_C)\circ e}$ extends $\overline{\phi\circ e}$, as required. $\square$

6 Consequences for realizability

In this section we draw out the consequences of our previous results in the case that most interests us. We shall suppose that $\mathbf{C}$ is a category of sets and functions and that $\overline{(\ )}$ is the underlying set functor. Thus $\overline{(\ )}$ is faithful. What we have in mind is that $\mathbf{C}$ is the category obtained from some form of typed partial applicative structure, as in Longley [14], but part of the game is to see how much of that structure we can reconstruct from properties of the resulting realizability category.

In section 3, we examined the case when a monoidal structure induced a product on $\mathcal{F}(\mathbf{C})$. In lemma 3.4 we showed that in this case we had a diagonal $d_C\colon C\longrightarrow C\otimes C$ and collection of maps into the unit $t_C\colon C\longrightarrow I$, satisfying certain properties. We have seen that these induce projections $\pi_0\colon X\otimes Y\longrightarrow X$ and $\pi_1\colon X\otimes Y\longrightarrow Y$, and a form of pairing $\langle a,b\rangle\colon z\longrightarrow X\otimes Y$. This pairing satisfies the beta laws:

$$\pi_0\circ\langle a,b\rangle = a \quad\text{and}\quad \pi_1\circ\langle a,b\rangle = b$$

but not necessarily the eta law

$$\langle\pi_0,\pi_1\rangle = \mathrm{id}\colon X\otimes Y\longrightarrow X\otimes Y$$

The result is that we have something which is almost, but not quite, a category of partial maps on a category with finite products. It is interesting to compare with the formalisms given in [18], and to check when the equations listed there are satisfied. It turns out that the transformations have the correct naturality properties, but equations whose domains are tensors $X \otimes Y$ are valid only when composed with the retraction on $X \otimes Y$. We can therefore obtain a category of partial maps by splitting suitable idempotents. Since idempotents split in **Ptl**, $\overline{(\;)}$ extends to the resulting category (though it is not obviously still faithful). Now

6.1 LEMMA *If in* $\mathbf{C}$, C *is a retract of* D

$$C \underset{i}{\overset{r}{\rightleftarrows}} D$$

then an object $f\colon X \longrightarrow \overline{C}$ *of* $\mathcal{F}(\mathbf{C})$ *is isomorphic to* $\bar{\imath} \circ f\colon X \longrightarrow \overline{D}$.

6.2 COROLLARY *If* $\overline{(\;)}\colon \mathbf{C} \longrightarrow \mathbf{Ptl}$ *and* $\mathbf{D}$ *is a category obtained from* C *by splitting idempotents, then* $\overline{(\;)}$ *extends to* $\mathbf{D}$, *and* $\mathcal{F}(\mathbf{D})$ *is equivalent to* $\mathcal{F}(\mathbf{C})$.

So this process does not affect the resulting category.

This means that if $\mathcal{F}(\mathbf{C})$ is a left exact category (or more exactly if it is lex and that structure is obtained from monoidal structure on $\mathbf{C}$), then $\mathbf{C}$ must already have interpretations of the combinators for pairing and unpairing satisfying similar properties to the pairing and unpairing in **Ptl**.

Suppose now, that $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ is locally cartesian closed. Then by corollary 5.7, $\mathbf{C}$ has weak partial evaluations. This means that for any pair C, D of objects of $\mathbf{C}$, there is an object which we can call $[C \rightharpoonup D]$ together with an evaluation map $\epsilon\colon [C \rightharpoonup D] \otimes C \longrightarrow D$. This generates an "application" in **Ptl**: $\bar{\epsilon} \circ \theta\colon \overline{[C \rightharpoonup D]} \times \overline{C} \longrightarrow \overline{D}$. We use this to construct a partial combinatory type structure in the sense of Longley [14].

The type world T is the set $\mathbf{C}_0$ of objects of $\mathbf{C}$, the binary product operation $C \times D$ is tensor product $C \otimes D$, and the arrow type is given by the weak partial evaluations $[C \rightharpoonup D]$. The associated family of sets is $(A_C | C \in \mathbf{C}_0) = (\overline{C} | C \in \mathbf{C}_0)$, and the application functions $\overline{[C \rightharpoonup D]} \times \overline{C} \longrightarrow \overline{D}$ are as above.

We also require s and k combinators, along with combinators for pairing and first and second projections. These are obtained by currying corresponding maps in $\mathbf{C}$. For example, the combinator $k \in \overline{[C \rightharpoonup D \rightharpoonup C]}$ is obtained

25 from $\pi_0\colon C \otimes D \longrightarrow C$. We first curry to get a map $k_1\colon C \longrightarrow [D \rightharpoonup C]$, and then again to get $k_2\colon I \longrightarrow [C \rightharpoonup D \rightharpoonup C]$, apply $\overline{(\ )}$ to get a (total) function $\overline{I} \longrightarrow \overline{[C \rightharpoonup D \rightharpoonup C]}$, and finally compose this with $\psi\colon 1 \longrightarrow \overline{I}$ to get k. The construction of s is similar, this time starting with

$$\begin{tikzcd}
(C \rightharpoonup D \rightharpoonup E) \otimes (C \rightharpoonup D) \otimes C \arrow[d, "\mathrm{id} \otimes \mathrm{id} \otimes d"] \\
(C \rightharpoonup D \rightharpoonup E) \otimes (C \rightharpoonup D) \otimes C \otimes C \arrow[d, "\sim"] \\
(C \rightharpoonup D \rightharpoonup E) \otimes C \otimes (C \rightharpoonup D) \otimes C \arrow[d, "\epsilon \otimes \epsilon"] \\
(D \rightharpoonup E) \otimes D \arrow[d, "\epsilon"] \\
E
\end{tikzcd}$$

Pairing and unpairing combinators are also obtained in this way, and satisfy the requisite equations. This can be seen from the following lemma.

6.3 LEMMA *Suppose $f\colon C \longrightarrow D$ is curried to give $F\colon I \longrightarrow [C \rightharpoonup D]$, then for all $c \in \overline{C}$, if $\overline{f}(c)$ is defined, then so is $(\overline{\epsilon} \circ \theta \circ (\overline{F} \times \mathrm{id}_{\overline{C}})(\psi, c)$, and they are equal.*

Proof $(\overline{\epsilon} \circ \theta \circ (\overline{F} \times \mathrm{id}_{\overline{C}})(\psi, c) = (\overline{\epsilon \circ (F \otimes \mathrm{id}_C)} \circ \theta)(\psi, c)$, and the result follows from corollary 5.7. $\square$

We thus have a partial combinatory type structure. This in turn generates a graph $\mathbf{C}'$ equipped with a graph morphism into $\mathbf{Ptl}$: the vertices are the same as the objects of $\mathbf{C}$, and the edges are the partial functions induced from the arrow types by application. It is irritating that $\mathbf{C}'$ is not necessarily a category, but it may fail to be closed under composition (we only know that the composite of two partial functions in the graph can be extended to a third). But we only need this, not full category structure, to define $\mathcal{F}(\mathbf{C}')$. Since any partial function obtained from $\mathbf{C}$ is extended by a partial function obtained from $\mathbf{C}'$, there is an embedding $\mathcal{F}(\mathbf{C}) \longrightarrow \mathcal{F}(\mathbf{C}')$. Unfortunately, this is not necessarily an equivalence. The problem is that there is no reason why a partial function obtained from $\mathbf{C}'$ should extend to one obtained from $\mathbf{C}$. Suppose, however, that $\mathbf{C}$ is concrete in the sense that every element of $\overline{C}$ is obtained from a morphism $I \longrightarrow C$. In this case every partial function in $\mathbf{C}'$ is already in $\mathbf{C}$, and the realizability structure obtained categorically is identical to that obtained from the partial combinatory type structure.

We now turn our attention to the case when $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ has a subobject classifier. In that case we have that $\mathbf{C}$ has a universal object V, and applying 6.1, we get that $\mathcal{F}(\mathbf{C})$ is equivalent to $\mathcal{F}(\mathbf{M})$ where $\mathbf{M}$ is the monoid of endomorphisms on V.

6.4 COROLLARY *If $\mathcal{F}(\mathbf{C})$ is a topos, then it is equivalent to the topos constructed using the monoid of endomorphisms of the universal object in* $\mathbf{C}$.

Putting these observations together, we can see that if $\mathcal{F}(\mathbf{C})_{\mathrm{ex}}$ is a topos, then, much as in Scott [21], $\overline{V}$ is a partial combinatory algebra, and if $\mathbf{C}$ is concrete, then the topos obtained is the conventional realizability topos from this algebra.

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