Some notes on equalities not present in the $\lambda\mu$-calculus.

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1 Introduction

Ever since Gentzen introduced his structural proof systems of natural deduction and sequent calculus it has been folklore that the sequent calculus is intended to give an account of provability in natural deduction. However the standard proofs of equivalence between the systems give more: a proof in the sequent calculus can be translated into a natural deduction proof, which we can think of as its intended semantics. This mapping is many-one and onto. It's now natural to ask about the connection between the reduction systems for proofs, cut elimination on the one hand, and normalisation on the other.

2 The intuitionistic case

We begin with the intuitionistic case, which is established and well known, in order to provide a framework. To keep things simple we work with the implicational fragments of Gentzen's LJ and NJ.

The relevant rules for sequent calculus are

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,(\to R) \qquad\qquad \frac{\Gamma \vdash A \quad \Gamma', B \vdash C}{\Gamma, \Gamma', A \to B \vdash C}\,(\to L)$$

$$\frac{\Gamma \vdash A \quad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\,(\mathrm{Cut})$$

There are also the rules of contraction, weakening and exchange, but these will not be (overtly) relevant to us.

We shall use the simply typed lambda calculus as a language for denoting natural deduction proofs.

A proof

$$\frac{\begin{array}{c}\mathcal{P}\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \to B}\,(\to R)$$

translates to a natural deduction proof

$$\frac{\begin{array}{c}\Gamma \quad [A]\\ \vdots\\ (\!(\mathcal{P})\!)\\ \vdots\\ B\end{array}}{A \to B}\,(\to I)$$

Here I've used $(\!(\mathcal{P})\!)$ for the translation of proof $\mathcal{P}$.

Syntactically we get

$$\Gamma \vdash \lambda a.[\![\mathcal{P}]\!] : A \to B$$

where I've used $[\![\mathcal{P}]\!]$ for the syntactic translation.

The other cases are more interesting:

$$\dfrac{\begin{matrix}\mathcal{P}\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\mathcal{P}'\\ \Gamma', B \vdash C\end{matrix}}{\Gamma, \Gamma', A \to B \vdash C}\ (\to L)$$

translates to

$$\begin{array}{ccc} & & \Gamma \\ & & \vdots \\ & & (\!(\mathcal{P})\!) \\ & & \vdots \\ \Gamma' & & \dfrac{A \quad A \to B}{B}\ (\to E) \\ & \vdots & \\ & (\!(\mathcal{P}')\!) & \\ & \vdots & \\ & C & \end{array}$$

or syntactically

$$\Gamma, \Gamma', f : A \to B \vdash [\![\mathcal{P}']\!][f([\![\mathcal{P}]\!])/b] : C$$

where $u[v/x]$ is the metalevel operation of substitution.

$$\dfrac{\begin{matrix}\mathcal{P}\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\mathcal{P}'\\ \Gamma', A \vdash B\end{matrix}}{\Gamma, \Gamma' \vdash B}\ (\mathrm{Cut})$$

translates to

$$\begin{array}{ccc} & & \Gamma \\ & & \vdots \\ & & (\!(\mathcal{P})\!) \\ & & \vdots \\ \Gamma' & & A \\ & \vdots & \\ & (\!(\mathcal{P}')\!) & \\ & \vdots & \\ & B & \end{array}$$

or syntactically

$$\Gamma, \Gamma' \vdash [\![\mathcal{P}']\!][[\![\mathcal{P}]\!]/a] : B$$

This brings us to the point where we can discuss the link between cut elimination and normalisation. The interesting case for us is when at least one of the cut formulae has not just been introduced. Of course, this can either be in the sequent on the left or on the right.

If the cut formula A in the sequent on the left has not just been introduced, then that sequent must be the consequence of either a structural rule, or a left rule. If the latter, then the proof is of the form

$$\dfrac{\dfrac{\begin{matrix}\mathcal{P}\\ \Gamma \vdash P\end{matrix} \qquad \begin{matrix}\mathcal{P}'\\ \Gamma', Q \vdash A\end{matrix}}{\Gamma, \Gamma', P \to Q \vdash A}\ (\to L) \qquad \begin{matrix}\mathcal{P}''\\ \Gamma'', A \vdash B\end{matrix}}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash B}\ (\mathrm{Cut})$$

This corresponds to a natural deduction proof

$$
\begin{array}{ccc}
\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ (\!(\mathcal{P})\!) \\ \vdots \\ P\end{array} \qquad P \to Q}{Q}\,(\to E) & \Gamma' & \\
\begin{array}{c}\vdots \\ (\!(\mathcal{P}')\!) \\ \vdots \\ A\end{array} & & \Gamma'' \\
 & \begin{array}{c}\vdots \\ (\!(\mathcal{P}'')\!) \\ \vdots \\ B\end{array} &
\end{array}
$$

and to a lambda term

$$\Gamma, \Gamma', \Gamma'', f : P \to Q \vdash [\![\mathcal{P}'']\!][[\![\mathcal{P}']\!][f([\![\mathcal{P}]\!])/q]/a]$$

The cut can be pushed up into the leftsubtree yielding

$$
\dfrac{\begin{array}{c}\mathcal{P} \\ \Gamma \vdash P\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{P}' \\ \Gamma', Q \vdash A\end{array} \qquad \begin{array}{c}\mathcal{P}'' \\ \Gamma'', A \vdash P\end{array}}{\Gamma', \Gamma'', Q \vdash B}\,(\mathrm{Cut})}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash B}\,(\to L)
$$

This prooftree yields the same natural deduction proof as previously, but an apparently slightly different lambda term:

$$\Gamma, \Gamma', \Gamma'', f : P \to Q \vdash [\![\mathcal{P}'']\!][[\![\mathcal{P}']\!]/a][f([\![\mathcal{P}]\!])/q]$$

These two terms are however equal by the substitution lemma.

In the other case, when the A on the right has not just been introduced, we can use either a left or right rule. For a right rule the proof looks like

$$
\dfrac{\begin{array}{c}\mathcal{P} \\ \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{P}' \\ \Gamma', A, B \vdash C\end{array}}{\Gamma', A \vdash B \to C}\,(\to R)}{\Gamma, \Gamma' \vdash B \to C}\,(\mathrm{Cut})
$$

This corresponds to the natural deduction proof:

$$
\dfrac{\begin{array}{ccc} & \begin{array}{c}\Gamma \\ \vdots \\ (\!(\mathcal{P})\!) \\ \vdots\end{array} & \\ \Gamma' & A & [B] \\ & \begin{array}{c}\vdots \\ (\!(\mathcal{P}')\!) \\ \vdots \\ C\end{array} & \end{array}}{B \to C}\,(\to I)
$$

Syntactically, we have

$$\Gamma, \Gamma' \vdash (\lambda b.\ [\![\mathcal{P}']\!])[[\![\mathcal{P}]\!]/a]$$

The cut can be pushed up into the right hand subtree, giving

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{P}\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{P}'\\ \Gamma', A, B \vdash C\end{array}}{\Gamma, \Gamma', B \vdash C}\,(\mathrm{Cut})}{\Gamma, \Gamma' \vdash B \to C}\,(\to R)$$

This again corresponds to the same natural deduction proof, but an apparently slightly different lambda term:

$$\Gamma, \Gamma' \vdash \lambda b.\ ([\![\mathcal{P}']\!][[\![\mathcal{P}]\!]/a])$$

However, since a and b are distinct, the two terms are in fact equal.

When the previous rule is a left rule then we have:

$$\dfrac{\begin{array}{c}\mathcal{P}\\ \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{P}'\\ \Gamma', A \vdash R\end{array} \qquad \begin{array}{c}\mathcal{P}''\\ \Gamma'', S \vdash B\end{array}}{\Gamma', \Gamma'', R \to S, A \vdash B}\,(\to L)}{\Gamma, \Gamma', \Gamma'', P \to Q, R \to S \vdash B}\,(\mathrm{Cut})$$

corresponding to the natural deduction proof:

$$\begin{array}{ccc}
 & \begin{array}{c}\Gamma\\ \vdots\\ (\!(\mathcal{P})\!)\\ \vdots\\ A\end{array} & \\
\begin{array}{c}\Gamma'\\ \vdots\\ (\!(\mathcal{P}')\!)\\ \vdots\\ R\end{array} & & \\
\end{array}$$

$$\dfrac{R \qquad\qquad R \to S}{S}\,(\to E) \qquad \Gamma''$$

$$\begin{array}{c}\vdots\\ (\!(\mathcal{P}'')\!)\\ \vdots\\ B\end{array}$$

Or else we have its minor variant:

$$\dfrac{\begin{array}{c}\mathcal{P}\\ \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{P}'\\ \Gamma' \vdash R\end{array} \qquad \begin{array}{c}\mathcal{P}''\\ \Gamma'', A, S \vdash B\end{array}}{\Gamma', \Gamma'', R \to S, A \vdash B}\,(\to L)}{\Gamma, \Gamma', \Gamma'', P \to Q, R \to S \vdash B}\,(\mathrm{Cut})$$

Again, the cuts can be pushed up into the right-hand subtrees, and again this leaves the natural deduction proofs unchanged.

There are sound intuitive reasons why these should come out the same. The interpretation of cut is to stick one proof on top of another. The other rules construct proofs by operations at the major formulae of the rules. In our case none of these formulae are near the cut formula. Therefore it does not matter whether we first construct a proof using the chosen proof rule, and then glue another bit on using cut, or do the gluing using cut first, before doing the rest of the assembly of the tree.

3 The classical case

The classical case is more complex. First, Gentzen's LK is a multiple conclusion logic. This generates more possibilities for trivial modifications of proofs. Second, while NJ is a definitive intuitionistic calculus of natural deduction, the classical NK is a hack with the double negation rule not corresponding to natural deduction's introduction and elimination pattern. Finally, instead of simply typed lambda calculus the analogous calculus is the $\lambda\mu$-calculus (or at least $\lambda\mu$-terms form a suitable collection to notate the constructions).

The LK proof rules are

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\,(\to R) \qquad\qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash C, \Delta'}{\Gamma, \Gamma', A \to B \vdash C, \Delta, \Delta'}\,(\to L)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash B, \Delta'}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,(\mathrm{Cut})$$

We consider the case when at least one of the cut formulae has not just been introduced. As before, there are two possibilities, either it is the A in $\Gamma \vdash A, \Delta$, a left occurrence, or it is the A in $\Gamma', A \vdash B, \Delta'$. Each of these cases splits according to whether the rule above the cut is a left or right rule, and in the event that it is a left rule, splits again according to which part of the proof the A comes from.

Case (RR) The proof is

$$\frac{\begin{array}{c}\mathcal{P}\\ \Gamma \vdash A, \Delta\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{P}'\\ \Gamma', A, P \vdash Q, \Delta'\end{array}}{\Gamma', A \vdash P \to Q, \Delta'}\,(\to R)}{\Gamma, \Gamma' \vdash P \to Q, \Delta, \Delta'}\,(\mathrm{Cut})$$

This is essentially an intuitionistic case.

Case (RL) This splits into two subcases:

$$\frac{\begin{array}{c}\mathcal{P}\\ \Gamma \vdash A, \Delta\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{P}'\\ \Gamma', A \vdash P, \Delta'\end{array} \qquad \begin{array}{c}\mathcal{P}''\\ \Gamma'', Q \vdash \Delta''\end{array}}{\Gamma', \Gamma'', P \to Q, A \vdash \Delta'}\,(\to L)}{\Gamma, \Gamma', P \to Q \vdash \Delta, \Delta'}\,(\mathrm{Cut})$$

and

$$\frac{\begin{array}{c}\mathcal{P}\\ \Gamma \vdash A, \Delta\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{P}'\\ \Gamma' \vdash P, \Delta'\end{array} \qquad \begin{array}{c}\mathcal{P}''\\ \Gamma'', A, Q \vdash \Delta''\end{array}}{\Gamma', \Gamma'', P \to Q, A \vdash \Delta'}\,(\to L)}{\Gamma, \Gamma', P \to Q \vdash \Delta, \Delta'}\,(\mathrm{Cut})$$

but again both have intuitionistic analogues.

Case (LR)

$$\frac{\dfrac{\begin{array}{c}\mathcal{P}\\ \Gamma, P \vdash A, Q, \Delta\end{array}}{\Gamma \vdash A, P \to Q, \Delta}\,(\to R) \qquad \begin{array}{c}\mathcal{P}'\\ \Gamma', A \vdash \Delta'\end{array}}{\Gamma, \Gamma' \vdash P \to Q, \Delta, \Delta'}\,(\mathrm{Cut})$$

This cannot arise intuitionistically, since we make essential use of the multiple conclusion nature of the logic. This use hides a corresponding switching of conclusions, which is made explicit in the resulting $\lambda\mu$-term.

Suppose $[\![\mathcal{P}]\!]$ yields a term with Q in the stoup, then $\lambda p.[\![\mathcal{P}]\!]$ yields a term with $P \to Q$ in the stoup. In order to apply cut we need to refocus to obtain a term with A in the stoup: $\mu\alpha.[\phi]\lambda p.[\![\mathcal{P}]\!]$. This gives us

$$[\![\mathcal{P}']\!][\mu\alpha.[\phi]\lambda p.[\![\mathcal{P}]\!]/a]$$

We can push this cut upwards to get

$$\dfrac{\dfrac{\dfrac{\mathcal{P}}{\Gamma, P \vdash A, Q, \Delta} \qquad \dfrac{\mathcal{P}'}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma', P \vdash Q, \Delta, \Delta'}\,(\text{Cut})}{\Gamma, \Gamma' \vdash P \to Q, \Delta, \Delta'}\,(\to R)$$

Here we must first refocus term $[\![\mathcal{P}]\!]$ as $\mu\alpha.[\kappa][\![\mathcal{P}]\!]$, and then substitute in $[\![\mathcal{P}']\!]$, before refocusing on Q and then lambda abstracting to obtain $\lambda p.\mu\kappa.[\delta'][\![\mathcal{P}']\!][\mu\alpha.[\kappa][\![\mathcal{P}]\!]/a]$. However, this term has $P \to Q$ in the stoup, whereas the previous term had some element of Δ', so we need refocus one of them (it does not matter which). If we refocus this one then we get

$$\mu\delta'.[\phi]\lambda p.\mu\kappa.[\delta'][\![\mathcal{P}']\!][\mu\alpha.[\kappa][\![\mathcal{P}]\!]/a]$$

Analogously to the intuitionistic case, the two terms "should" denote identical natural deduction proofs. There is therefore an argument for their equality. Generalising to put variables in place of the proof identifiers, we get

$$\mu\delta'.[\phi]\lambda p.\mu\kappa.[\delta']v[\mu\alpha.[\kappa]u/a] = v[\mu\alpha.[\phi]\lambda p.u/a]$$

or alternatively

$$\lambda p.\mu\kappa.[\delta']v[\mu\alpha.[\kappa]u/a] = \mu\phi.[\delta']v[\mu\alpha.[\phi]\lambda p.u/a]$$

These are not valid equations in $\lambda\mu$, though when one takes the second (functional) form, and applies it to an argument, then it becomes valid. This is therefore a form of η equality.

It is therefore intriguing that this equation is valid in Ong's form of $\lambda\mu$ where it follows from his (ζ) law (C-H.L. Ong: A semantic view of classical proofs).

Case (LL)

$$\dfrac{\dfrac{\dfrac{\mathcal{P}}{\Gamma \vdash P, A, \Delta} \qquad \dfrac{\mathcal{P}'}{\Gamma', Q \vdash \Delta'}}{\Gamma, \Gamma', P \to Q \vdash A, \Delta, \Delta'}\,(\to L) \qquad \dfrac{\mathcal{P}''}{\Gamma'', A \vdash \Delta''}}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash \Delta, \Delta', \Delta''}\,(\text{Cut})$$

for which the corresponding $\lambda\mu$-term is (supposing in the first term that P is in the stoup)

$$[\![\mathcal{P}'']\!][\mu\alpha.[\delta'][\![\mathcal{P}']\!][f([\![\mathcal{P}]\!])/q]/a]$$

This cut can be pushed up to give:

$$\dfrac{\dfrac{\dfrac{\mathcal{P}}{\Gamma \vdash P, A, \Delta} \qquad \dfrac{\mathcal{P}''}{\Gamma'', A \vdash \Delta''}}{\Gamma, \Gamma'' \vdash P, \Delta, \Delta''}\,(\text{Cut}) \qquad \dfrac{\mathcal{P}'}{\Gamma', Q \vdash \Delta'}}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash \Delta, \Delta', \Delta''}\,(\to L)$$

with corresponding $\lambda\mu$-term

$$\mu\delta''.[\delta'][\![\mathcal{P}']\!][f(\mu\pi.[\delta''][\![\mathcal{P}'']\!][\mu\alpha.[\pi][\![\mathcal{P}]\!]/a])/q]$$

Again we can argue that these terms represent the same natural deduction proofs. This suggests an equation

$$w[\mu\alpha.[\delta']v[f(u)/q]/a] = \mu\delta''.[\delta']v[f(\mu\pi.[\delta'']w[\mu\alpha.[\pi]u/a])/q]$$

or equivalently

$$\mu\delta'.[\delta'']w[\mu\alpha.[\delta']v[f(u)/q]/a] = v[f(\mu\pi.[\delta'']w[\mu\alpha.[\pi]u/a])/q]$$

The alternative (essentially intuitionistic) proof

$$\dfrac{\dfrac{\begin{matrix}\mathcal{P} \\ \Gamma \vdash P, \Delta\end{matrix} \qquad \begin{matrix}\mathcal{P}' \\ \Gamma', Q \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', P \to Q \vdash A, \Delta, \Delta'}\,(\to L) \qquad \begin{matrix}\mathcal{P}'' \\ \Gamma'', A \vdash \Delta''\end{matrix}}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash \Delta, \Delta', \Delta''}\,(\mathrm{Cut})$$

yields

$$[\![\mathcal{P}'']\!][[\![\mathcal{P}']\!][f([\![\mathcal{P}]\!])/p]/a]$$

When we push the cut to

$$\dfrac{\begin{matrix}\mathcal{P} \\ \Gamma \vdash P, \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{P}' \\ \Gamma', Q \vdash A, \Delta'\end{matrix} \qquad \begin{matrix}\mathcal{P}'' \\ \Gamma'', A \vdash \Delta''\end{matrix}}{\Gamma', \Gamma'', Q \vdash \Delta', \Delta''}\,(\mathrm{Cut})}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash \Delta, \Delta', \Delta''}\,(\to L)$$

we get

$$[\![\mathcal{P}'']\!][[\![\mathcal{P}']\!]/a][f([\![\mathcal{P}]\!])/p]$$

These are equivalent by standard properties of substitution.